

Cells with many facets in a Poisson hyperplane tessellation

Gilles Bonnet* Pierre Calka† Matthias Reitzner‡

Abstract

Let Z be the typical cell of a stationary Poisson hyperplane tessellation in \mathbb{R}^d . The distribution of the number of facets $f(Z)$ of the typical cell is investigated. It is shown, that under a *well-spread* condition on the directional distribution, the quantity $n^{\frac{2}{d-1}} \sqrt[n]{\mathbb{P}(f(Z) = n)}$ is bounded from above and from below. When $f(Z)$ is large, the isoperimetric ratio of Z is bounded away from zero with high probability.

These results rely on one hand on the Complementary Theorem which provides a precise decomposition of the distribution of Z and on the other hand on several geometric estimates related to the approximation of polytopes by polytopes with fewer facets.

From the asymptotics of the distribution of $f(Z)$, tail estimates for the so-called Φ content of Z are derived as well as results on the conditional distribution of Z when its Φ content is large.

Keywords. Poisson hyperplane tessellation, random polytopes, typical cell, directional distribution, Complementary Theorem, D.G. Kendall's problem, shape distribution

MSC. 60D05, 52A22

1 Introduction

One of the classical models in stochastic geometry to generate a random mosaic is the construction via a Poisson hyperplane process. A Poisson hyperplane process consists of countably many random hyperplanes in \mathbb{R}^d chosen in such a way, that their distribution is translation invariant, the distribution of the direction of the

*Institut für Mathematik, Universität Osnabrück, Albrechtstr. 28a, 49076 Osnabrück, Germany. Email: gilles.bonnet@uni-osnabrueck.de

†Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, Avenue de l'Université, BP.12, Technopôle du Madrillet, F76801 Saint-Etienne-duRouvray France. Email: pierre.calka@univ-rouen.fr

‡Institut für Mathematik, Universität Osnabrück, Albrechtstr. 28a, 49076 Osnabrück, Germany. Email: matthias.reitzner@uni-osnabrueck.de

hyperplanes follows a directional distribution φ , and the number of hyperplanes hitting an arbitrary convex set K is Poisson distributed.

Such a Poisson hyperplane process tessellates \mathbb{R}^d into countably many convex polytopes, the tiles of the mosaic. The distribution of a tile chosen at random is the distribution of the so-called *typical cell* Z , a random polytope.

The typical cell has been investigated intensively in the past decades, numerous papers have been dedicated to describe quantities associated with this cell, for example volume, surface area, mean width, number of facets, etc. The expected number of facets $f(Z)$ of the typical cell and the expected volume $V_d(Z)$ are known, see e.g. the first works due to Miles [17, 18] and Matheron [15] as well as Chapter 10 from the seminal book of Schneider and Weil [22] and the survey [2].

But in almost all cases the distribution of these quantities is out of reach, and even good approximations are extremely difficult and unknown so far. Our main theorem fills this gap for the number of facets of Z , giving precise asymptotics for the tails of the distribution.

Theorem 1.1. *There exists a constant $c_1 > 0$, depending on φ , such that for $n \geq d + 1$,*

$$\mathbb{P}(f(Z) = n) < c_1^n n^{-\frac{2n}{d-1}}.$$

Furthermore, there exists an integer n_φ such that $\mathbb{P}(f(Z) = n)$ is either vanishing or strictly decreasing for $n \geq n_\varphi$.

Here and in the sequel, c_i will denote a positive constant which depends on dimension d . It will be specified when it depends on φ or another parameter.

It is clear that in general there is no matching lower bound, for example if the directions of the hyperplane process are concentrated on a finite set. We prove that, if the directional distribution satisfies a mild condition, we have lower bounds of the same order in n as the upper bound above. In the following, we call φ *well spread* if there exists a cap on the unit sphere where φ is bounded from below by a multiple of the surface area measure.

Theorem 1.2. *Assume that φ is well spread. Then there exists a constant $c_2 > 0$, depending on φ , such that for $n \geq d + 1$,*

$$\mathbb{P}(f(Z) = n) > c_2^n n^{-\frac{2n}{d-1}}.$$

The occurring constant will be made more explicit in Section 4, in particular its dependence on the directional distribution φ of η .

Maybe a simple conjecture for the distribution of the number of facets $f(Z)$ of the typical cell of a Poisson hyperplane tessellation would have been the Poisson distribution. Yet our theorem disproves this, as a more intricate conjecture we state that $f(Z)$ follows a Compound Poisson distribution.

Theorems 1.1 and 1.2 prove the asymptotic expansion

$$\ln \mathbb{P}(f(Z) = n) = -\frac{2}{d-1} n \ln n + \Theta(n)$$

as $n \rightarrow \infty$ where the implicit constants in the error term $\Theta(n)$ are strictly positive. We pose it as an open problem whether

$$\ln \mathbb{P}(f(Z) = n) = -\frac{2}{d-1} n \ln n + cn + \Theta(\ln n).$$

Support for this conjecture comes from the planar case where Calka and Hilhorst [4] stated a more precise result if φ is rotation invariant. In [7], Hilhorst investigates the similar case of the typical cell of a Poisson-Voronoi tessellation and provides heuristics for getting an analogous asymptotic expansion for the probability that the typical Poisson-Voronoi cell has n facets. He obtains as a first term $-\frac{2}{d-1}n \ln(n)$, as in the present paper, but does not make the second term fully explicit (indeed, the constant c_d introduced in (2.10) therein is unknown). In particular, the approximation result provided by Lemma 3.3 matches to some extent the statement (2.10) in [7], i.e. *many* of the n facets of Z lie in an annulus with thickness of order $n^{-\frac{2}{d-1}}$ multiplied by a size functional of Z . This suggests that improving Theorems 1.1 and 1.2 requires new ingredients and notably a substantial improvement of Lemma 3.3.

In the following our aim is to show that cells with many facets are far away from any lower dimensional convex body. To do this we measure the distance from the ball using the isoperimetric ratio of a convex set K . Denote by V_i the i -th intrinsic volume (see Section 2.1 for the definition). For any $1 \leq i < j \leq d$ we call $V_j(K)^{1/j} V_i(K)^{-1/i}$ the (i, j) -isoperimetric ratio of K . The isoperimetric inequality says that this ratio is maximized precisely for balls. On the other hand when the isoperimetric ratio of K vanishes, K must be lower dimensional.

The next theorem shows that the isoperimetric ratio $V_j(Z)^{1/j} V_i(Z)^{-1/i}$ of the typical cell is bounded away from zero with high probability if the cell has many facets. Hence cells with many facets cannot be too elongated.

Theorem 1.3. *Assume that φ is well spread and that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. For any $\delta \in (0, 1)$, there exist constants ϵ and n_0 , depending on φ, i, j and δ such that*

$$\mathbb{P} \left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \mid f(Z) = n \right) \leq \delta^n$$

for all $n \geq n_0$.

To describe the distribution of the typical cell Z we need the notion of the Φ -content of a convex body K . The Φ -content measures in a certain sense the size of the convex set depending on the directional distribution φ of the hyperplane tessellation. It is given by

$$\Phi(K) := \int_{\mathbb{S}^{d-1}} h(K, \mathbf{u}) \, d\varphi(\mathbf{u}),$$

where $h(K, \mathbf{u}) := \max\{\langle \mathbf{x}, \mathbf{u} \rangle \mid \mathbf{x} \in K\}$ is the value of the support function of K at \mathbf{u} . That the Φ content is an important quantity of a Poisson hyperplane process is immediately clear since the number of hyperplanes hitting an arbitrary convex set K is Poisson distributed with parameter $\gamma\Phi(K)$. The real number $\gamma > 0$ is the intensity of the hyperplane process. In the important case where φ is a constant and hence the directional distribution is the uniform distribution on \mathbb{S}^{d-1} , the Φ -content of a convex set K is just the well known mean width $V_1(K)$ of K up to a constant. For more information we refer to Section 2.1.

Again the distribution of $\Phi(Z)$ is unknown, and in this case even the expectation is out of reach. Here we succeed in computing the tail behaviour of the size-functional $\Phi(Z)$.

Theorem 1.4. *There exist constants $c_3 > c_4 > 0$ and $c_5 > 0$ depending on φ , such that the following holds. For $a > 0$, we have*

$$\mathbb{P}(\Phi(Z) > a) < \exp \left\{ -\gamma a + c_3(\gamma a)^{\frac{d-1}{d+1}} \right\}.$$

Assume that φ is well spread. Then, for $a > \gamma^{-1}c_5$, we also have

$$\mathbb{P}(\Phi(Z) > a) > \exp \left\{ -\gamma a + c_4(\gamma a)^{\frac{d-1}{d+1}} \right\}.$$

Again we can use our bounds on the distribution of the Φ -content of the typical cell to show that big cells are not too elongated.

Theorem 1.5. *Assume that φ is well spread and that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. Then for any $\epsilon > 0$ sufficiently small we have*

$$\lim_{a \rightarrow \infty} \mathbb{P} \left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \mid \Phi(Z) > a \right) = 0.$$

It is a long standing general conjecture that extremal cells of Poisson hyperplane mosaics converge to a limit shape. This question is known as Kendall's problem and has attained great interest with a large number of important contributions, see [14] [10] [11] [12] [13] and the surveys [2] [8] [9] [3]. For a precise definition of the *shape* $\mathfrak{s}(Z)$, of the cell Z we refer to Section 2.1. It turned out that many size functionals allow positive solutions of Kendall's problem, but the first intrinsic volume, the Φ -content and the number of facets resisted all attempts so far. In the contrary, Hug and Schneider [11, Thm. 4] gave an example where the shape of the cell containing the origin, i.e. the zero cell, under the condition that it has a big Φ -content does *not* concentrate.

Theorems 1.3 and 1.5 are first attempts to close the existing gaps. They show that the shape $\mathfrak{s}(Z)$ of the typical cell cannot be too elongated if either the number of facets or the Φ -content is large. At a first glance they seem to be in conflict with the example given by Hug and Schneider. Yet for their example they used a measure φ which is concentrated on finitely many points and thus not well spread. Next we prove that their theorem holds when the zero cell is replaced by the typical cell.

Theorem 1.6. Assume φ is concentrated on a finite number of points, $f(Z) \leq n_{\max}$ with probability one. Then there is a limiting shape distribution,

$$\lim_{a \rightarrow \infty} \mathbb{P}(\mathfrak{s}(Z) \in S | \Phi(Z) \geq a) = \mathbb{P}(\mathfrak{s}(Z) \in S | f(Z) = n_{\max}).$$

Note that when conditioning on the number of facets $f(Z)$, the shape $\mathfrak{s}(Z)$ of the typical cell is independent of the size $\Phi(Z)$ (see the Complementary Theorem 2.2) and is given explicitly in Theorem 2.2.

The paper is organized as follows. In Section 2, we fix the general setting and introduce the so-called Complementary Theorem, which is a practical disintegration of the distribution of the typical cell Z . This is the fundamental probabilistic tool for showing our main results. In Section 3, we provide the required geometric ingredients which deal with the approximation of polytopes by polytopes with fewer facets. Section 4 is devoted to the proof of the main results about cells with many facets, i.e. Theorems 1.1, 1.2 and 1.3. Finally, in Section 5, we prove Theorems 1.4, 1.5 and 1.6 which deal with the *big cells*, i.e. with a large Φ content.

2 Notations and the Complementary Theorem

2.1 Setting and notations

As standard references to the following material from convex geometry we refer to the books by Schneider [21] and Gruber [6]

We work in a d -dimensional Euclidean vector space \mathbb{R}^d , $d \geq 2$, with scalar product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and origin \mathbf{o} . We denote by $B(\mathbf{x}, r)$ the closed ball and by $S(\mathbf{x}, r) = \partial B(\mathbf{x}, r)$ the sphere with center \mathbf{x} and radius r , by $B^d = B(\mathbf{o}, 1)$ the unit ball and by $\mathbb{S}^{d-1} = \partial B^d$ the unit sphere. Let \mathcal{H} be the space of affine hyperplanes in \mathbb{R}^d with its usual topology and Borel structure. Every hyperplane $H \in \mathcal{H}$ has a unique representation

$$H(\mathbf{u}, t) := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{u} \rangle = t\}.$$

with $\mathbf{u} \in \mathbb{S}^{d-1}$ and $t > 0$. For a given hyperplane $H \in \mathcal{H}$, we write H^- , resp. H^+ , for the closed halfspace with boundary H which contains, resp. excludes the origin,

$$H(\mathbf{u}, t)^- = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{u} \rangle \leq t\} \quad \text{and} \quad H(\mathbf{u}, t)^+ = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{u} \rangle \geq t\}.$$

We denote by $\tilde{\mathcal{H}} := \mathcal{H} \times \{\pm 1\}$ the space of halfspaces.

Let \mathcal{K} be the set of convex bodies (compact convex sets of \mathbb{R}^d with non-empty interior) and denote by K° the relative interior of a set $K \in \mathcal{K}$. We write \mathcal{P} for the set of all polytopes, and $f(P)$ for the number of facets of a polytope $P \in \mathcal{P}$. Denote by $\mathcal{P}_n = \{P \in \mathcal{P} | f(P) = n\}$ the set of n -topes, hence $\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{K}$. For any $t > 0$, and $K, L \in \mathcal{K}$ we define

$$tK := \{t\mathbf{x} : \mathbf{x} \in K\}, \quad K + L := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in K, \mathbf{y} \in L\}$$

where the latter is the Minkowski sum of K and L .

The sets \mathcal{K} , \mathcal{P} , and \mathcal{P}_n are equipped with the Hausdorff distance d_H ,

$$d_H(K, L) = \min\{r: K \subset L + rB^d, L \subset K + rB^d\},$$

and with the associated topology and Borel structure.

Steiner's formula says that the volume of the Minkowski sum of $K \in \mathcal{K}$ and a ball of radius r is a polynomial in r ,

$$V_d(K + rB^d) = \sum_{i=0}^d \kappa_i V_{d-i}(K) r^i$$

where κ_i denotes the volume of the i -dimensional unit ball, and $V_i(K)$ is the i -th *intrinsic volume* of K . E.g., V_d is the usual volume, $2V_{d-1}$ the surface area and V_1 a multiple of the mean width of K . Steiner's formula can be generalized to all intrinsic volumes,

$$V_j(K + rB^d) = \sum_{i=0}^j \binom{d-j+i}{i} \frac{\kappa_{d-j+i}}{\kappa_{d-j}} V_{j-i}(K) r^i. \quad (2.1)$$

The isoperimetric inequality tells us that for $1 \leq i < j \leq d$ the isoperimetric ratio is bounded,

$$\frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} \leq \frac{\kappa_j^{\frac{1}{j}}}{\kappa_i^{\frac{1}{i}}} \quad (2.2)$$

with equality if and only if K is a ball. The ratio is well defined if the dimension of K is at least i and equals zero if and only if the dimension is at most $(j-1)$. In Theorems 1.3 and 1.5 we prove that with high probability the isoperimetric ratio of the cells with many facets is bounded away from zero, thus proving that it is not too close to a $(j-1)$ -dimensional convex body.

Let η be a stationary Poisson hyperplane process in \mathbb{R}^d , that is a Poisson point process in the space \mathcal{H} invariant under translation. We often identify a simple counting measure with its support, so that for any set $\mathcal{A} \subset \mathcal{H}$, both notations $\eta(\mathcal{A})$ and $|\eta \cap \mathcal{A}|$ denote the number of elements of η in \mathcal{A} .

Since η is stationary, its intensity measure $\mathbb{E}\eta(\cdot)$ decomposes into Lebesgue measure and an even probability measure φ on \mathbb{S}^{d-1} ,

$$\mathbb{E}\eta(\cdot) = \gamma\mu(\cdot) := \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}(H(\mathbf{u}, t) \in \cdot) dt d\varphi(\mathbf{u}), \quad (2.3)$$

where $\gamma > 0$ and μ is a measure on \mathcal{H} . We call γ the intensity and φ the directional distribution of the hyperplane process η . We assume that the support of φ is not contained in a great circle of \mathbb{S}^{d-1} . When φ is the normalized surface area measure on \mathbb{S}^{d-1} , we say that η is isotropic.

The closure of each of the connected components of the complement of the union $\bigcup_{H \in \eta} H$ is almost surely a polytope (because the support of φ is not contained in a great circle). These polytopes are the cells of the Poisson hyperplane mosaic X induced by η . We can see X as a point process in \mathcal{P} . To describe the distribution of X we need the notion of the Φ -content of a convex body K which is given by

$$\Phi(K) := \int_{\mathbb{S}^{d-1}} h(K, \mathbf{u}) \, d\varphi(\mathbf{u}),$$

where $h(K, \mathbf{u}) := \max\{\langle \mathbf{x}, \mathbf{u} \rangle : \mathbf{x} \in K\}$ is the value of the support function of K at \mathbf{u} . In the important case when φ is a constant and hence the directional distribution is the uniform distribution on \mathbb{S}^{d-1} , the Φ -content of a convex set K is up to a constant just the first intrinsic volume $V_1(K)$.

We will have to replace bounds expressed in terms of the first intrinsic volume $V_1(K)$ by bounds expressed in terms of Φ . Because V_1 and Φ are homogeneous of degree one, and Φ is continuous and strictly positive on the compact set $\{\text{convex compact sets } K \subset \mathbb{R}^d : V_1(K) = 1 \text{ and } \mathbf{o} \in K\}$, we see that $0 < c_\Phi := \sup_{K \in \mathcal{K}} V_1(K)/\Phi(K) < \infty$. Thus for all $K \in \mathcal{K}$,

$$V_1(K) \leq c_\Phi \Phi(K). \quad (2.4)$$

The definition of Φ is motivated by the fact that, for any $K \in \mathcal{K}$,

$$\mathbb{P}(\eta \cap K = \emptyset) = e^{-\gamma \Phi(K)}.$$

Note that Φ is homogeneous of degree 1 and translation invariant, $\Phi(tK + \mathbf{x}) = t\Phi(K)$ for any $K \in \mathcal{K}$, $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^d$. In the special case where η is isotropic, 2Φ is the so called mean width of K . Note that $K \in \mathcal{K}$ implies that $\Phi(K) > 0$ since K contains at least 2 points. For a set $\mathcal{X} \subset \mathcal{K}$ of convex bodies we define

$$\mathcal{X}_\Phi := \{K \in \mathcal{X} : \Phi(K) = 1\} \subset \mathcal{X}.$$

Let $\mathbf{c} : \mathcal{K} \rightarrow \mathbb{R}^d$ be a *center function*, i.e. a measurable map compatible with translations and homogeneous under the scale action:

$$\mathbf{c}(tK + \mathbf{x}) = t\mathbf{c}(K) + \mathbf{x}$$

for any $t \in (0, \infty)$ and any $\mathbf{x} \in \mathbb{R}^d$. For example, \mathbf{c} can be the center of mass. In this paper we assume that $\mathbf{c}(K) \in K$ for every $K \in \mathcal{K}$ and that \mathbf{c} is 1-Lipschitz, i.e. $\|\mathbf{c}(K) - \mathbf{c}(L)\| \leq d_H(K, L)$ for $K, L \in \mathcal{K}$. For a set $\mathcal{X} \subset \mathcal{K}$ of convex bodies we define

$$\mathcal{X}_\mathbf{c} := \{K \in \mathcal{X} : \mathbf{c}(K) = \mathbf{o}\} \subset \mathcal{X}.$$

In particular, $\mathcal{P}_\mathbf{c}$ denotes the set of polytopes with center at the origin. Due to the natural homeomorphism

$$\begin{aligned} \mathcal{P} &\rightarrow \mathbb{R}^d \times \mathcal{P}_\mathbf{c} \\ P &\mapsto (\mathbf{c}(P), P - \mathbf{c}(P)), \end{aligned}$$

we will consider from now on X as a germ-grain process in \mathbb{R}^d with grain space \mathcal{P}_c . Since η is stationary, this is also the case for X . That implies the existence of a probability measure \mathbb{Q} on \mathcal{P}_c such that the intensity measure of the germ-grain process X decomposes into \mathbb{Q} and Lebesgue measure λ_d ,

$$\mathbb{E} X(\{P - \mathbf{c}(P) \in C, \mathbf{c}(P) \in A\}) = \gamma^{(d)} \lambda_d(A) \mathbb{Q}(C) \quad (2.5)$$

for $C \subset \mathcal{P}_c$. We call \mathbb{Q} the grain distribution, and the constant $\gamma^{(d)} = \mathbb{E} X(\{P \in \mathcal{P}, \mathbf{c}(P) \in [0, 1]^d\})$ the intensity of X . It is easy to see that $\gamma^{(d)}$ is a multiple of γ^d , where γ is the intensity of the Poisson hyperplane process. A random centred polytope $Z \in \mathcal{P}_c$ with distribution \mathbb{Q} is called typical cell of X .

For $K \in \mathcal{K}$, we define its *shape* to be

$$\mathfrak{s}(K) = \frac{1}{\Phi(K)}(K - \mathbf{c}(K)).$$

We have that \mathfrak{s} is translation and scale invariant, i.e. for any $K \in \mathcal{K}$, $t \in (0, \infty)$ and any $\mathbf{x} \in \mathbb{R}^d$, we have

$$\mathfrak{s}(tK + \mathbf{x}) = \mathfrak{s}(K).$$

We want to point out that in this paper the shape is not rotation invariant. We call the set $\mathcal{K}_{c,\Phi} = \mathfrak{s}(\mathcal{K})$ the *shape space*. Similarly we call $\mathcal{P}_{c,\Phi} = \mathfrak{s}(\mathcal{P})$ the *shape space of polytopes*, and $\mathcal{P}_{n,c,\Phi} = \mathfrak{s}(\mathcal{P}_n)$ the *shape space of n -topes*. Note that, $\mathcal{K}_{c,\Phi}$, $\mathcal{P}_{c,\Phi}$ and $\mathcal{P}_{n,c,\Phi}$ are compact spaces.

We have the following natural homeomorphism

$$\begin{aligned} \mathfrak{h} : \mathcal{K} &\rightarrow \mathbb{R}^d \times (0, \infty) \times \mathcal{K}_{c,\Phi} \\ K &\mapsto (\mathbf{c}(K), \Phi(K), \mathfrak{s}(K)). \end{aligned}$$

Restricting the domain of \mathfrak{h} to \mathcal{P}_n or $\mathcal{P}_{n,c}$ induces the homeomorphisms

$$\mathfrak{h}_n : P \mapsto (\mathbf{c}(P), \Phi(P), \mathfrak{s}(P)), \quad \text{and} \quad \mathfrak{h}_{n,c} : P \mapsto (\Phi(P), \mathfrak{s}(P)).$$

The measure μ given in (2.3) is by definition homogeneous and translation invariant. It gives rise to the measure

$$\tilde{\mu} := \mu \otimes \delta$$

on the set of halfspaces $\tilde{\mathcal{H}}$, where δ is the counting measure on $\{\pm 1\}$. This in turn induces naturally the measure μ_n on \mathcal{P}_n via

$$\mu_n(D) := \frac{1}{n!} \int_{\tilde{\mathcal{H}}^n} \mathbb{1} \left(\bigcap_{i=1}^n H_i^{\epsilon_i} \in D \right) d\tilde{\mu}^n(\mathbf{H}^\epsilon), \quad (2.6)$$

where $\tilde{\mu}^n := \tilde{\mu} \otimes \cdots \otimes \tilde{\mu}$ denotes the product measure and $\mathbf{H}^\epsilon := (H_1^{\epsilon_1}, \dots, H_n^{\epsilon_n})$. Note that, since $D \subset \mathcal{P}_n$, the integrand $\mathbb{1}(\bigcap_{i=1}^n H_i^{\epsilon_i} \in D)$ above is equal to 0 as soon as the intersection $\bigcap_{i=1}^n H_i^{\epsilon_i}$ is not a polytope with n facets.

Let $A \subset \mathbb{R}^d$ and $C \subset \mathcal{P}_{n,\mathfrak{c},\Phi}$ be Borel sets and $b > 0$. Because the measure μ_n is homogeneous of degree n and translation invariant we obtain

$$\begin{aligned}\mathfrak{h}_n(\mu_n)(A \times (0, b) \times C) &= b^n \mathfrak{h}_n(\mu_n)(b^{-1}A \times (0, 1) \times C) \\ &= \lambda_d(b^{-1}A) b^n \mathfrak{h}_n(\mu_n)([0, 1]^d \times (0, 1) \times C) \\ &= \lambda_d(A) b^{n-d} \mathfrak{h}_n(\mu_n)([0, 1]^d \times (0, 1) \times C).\end{aligned}$$

For $C = \mathcal{P}_n$ this immediately gives the following useful lemma. Here and in the following for abbreviation we put $P_{[n]} = \bigcap_{i \in I} H_i^{\epsilon_i}$ for $\mathbf{H}^\epsilon := (H_1^{\epsilon_1}, \dots, H_n^{\epsilon_n})$.

Lemma 2.1. *For any $b > 0$ and any Borel set $A \subset \mathbb{R}^d$ we have*

$$\begin{aligned}\int_{\tilde{\mathcal{H}}^n} \mathbb{1}(\mathfrak{c}(P_{[n]}) \in A) \mathbb{1}(\Phi(P_{[n]}) < b) \mathbb{1}(P_{[n]} \in \mathcal{P}_n) d\tilde{\mu}^n(\mathbf{H}^\epsilon) \\ = \lambda_d(A) b^{n-d} \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(\mathfrak{c}(P_{[n]}) \in [0, 1]^d) \mathbb{1}(\Phi(P_{[n]}) < 1) \mathbb{1}(P_{[n]} \in \mathcal{P}_n) d\tilde{\mu}^n(\mathbf{H}^\epsilon).\end{aligned}$$

To simplify our notation we introduce on $\mathcal{P}_{n,\mathfrak{c},\Phi}$ the normalized push forward measure

$$\mu_{n,\mathfrak{c},\Phi}(\cdot) = \mathfrak{h}_n(\mu_n)([0, 1]^d \times (0, 1) \times \cdot) = \mu_n(\mathfrak{h}_n^{-1}([0, 1]^d \times (0, 1) \times \cdot))$$

and on \mathbb{R}_+ the measure

$$\lambda_1^{(n)}(\cdot) = \int_0^\infty \mathbb{1}(t \in \cdot) n t^{n-1} dt,$$

which is homogeneous of degree n . With these notations the pushforward measure $\mathfrak{h}_n(\mu_n)$ splits into the following product of measures:

$$\mathfrak{h}_n(\mu_n) = \lambda_d \otimes \lambda_1^{(n-d)} \otimes \mu_{n,\mathfrak{c},\Phi}. \quad (2.7)$$

Because $\mu_{n,\mathfrak{c},\Phi}(\mathcal{P}_{n,\mathfrak{c},\Phi})$ is finite, $\mu_{n,\mathfrak{c},\Phi}(\cdot)/\mu_{n,\mathfrak{c},\Phi}(\mathcal{P}_{n,\mathfrak{c},\Phi})$ defines a probability measure on $\mathcal{P}_{n,\mathfrak{c},\Phi}$.

That all the measures mentioned in this section are connected seems already obvious. This will be made precise in the next section.

2.2 Complementary Theorem

The essential backbone of our paper is the Complementary Theorem. Similar results have been proved before, see e.g. Miles [16], Møller and Zuyev [19] and Cowan [5], but for our purposes we need a very detailed description which we could not find in the literature. Therefore and for the sake of completeness we state it here explicitly and give a proof.

Theorem 2.2. *Let $n \geq d + 1$ be an integer.*

1. *For any Borel set $S \in \mathcal{P}_{n,c,\Phi}$ of shapes we have*

$$\begin{aligned} \mathbb{P}(f(Z) = n, \mathfrak{s}(Z) \in S) \\ = \frac{\gamma^d}{\gamma^{(d)}} (n-d)! \int_{\mathcal{P}_n} \mathbb{1}(\mathfrak{c}(P) \in [0, 1]^d) \mathbb{1}(\Phi(P) < 1) \mathbb{1}(\mathfrak{s}(P) \in S) d\mu_n(P). \end{aligned} \quad (2.8)$$

2. **(Complementary Theorem)** *If we condition the typical cell Z to have n facets, then*

- (a) $\Phi(Z)$ and $\mathfrak{s}(Z)$ are independent random variables,
- (b) $\Phi(Z)$ is $\Gamma_{\gamma, n-d}$ distributed, and
- (c) $\mathfrak{s}(Z)$ has probability distribution $\mu_{n,c,\Phi}(\cdot) / \mu_{n,c,\Phi}(\mathcal{P}_{n,c,\Phi})$.

2.3 Proof of Theorem 2.2

The number of cells of the mosaic X in a subset $D \subset \mathcal{P}_n$ is

$$X(D) = \frac{1}{n!} \sum_{H_1, \dots, H_n \in \eta_{\neq}^n} \sum_{\epsilon \in \{\pm 1\}^n} \mathbb{1} \left(\bigcap_{i=1}^n H_i^{\epsilon_i} \in D \right) \mathbb{1} \left(\eta \cap \left(\bigcap_{i=1}^n H_i^{\epsilon_i} \right)^o = \emptyset \right),$$

because there are $n!$ possibilities of ordering a list of n different halfspaces. The Slivnyak-Mecke formula, see e.g. [SchneiderWeil08 p.68, Corollary 3.2.3] gives for $P_{[n]} = \bigcap_{i=1}^n H_i^{\epsilon_i}$ that

$$\begin{aligned} \mathbb{E}X(D) &= \frac{\gamma^n}{n!} \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in D) \mathbb{P}(\eta \cap P_{[n]}^o = \emptyset) d\tilde{\mu}^n(\mathbf{H}^\epsilon) \\ &= \gamma^n \int_{\mathcal{P}_n} \mathbb{1}(P \in D) \mathbb{P}(\eta \cap P^o = \emptyset) d\mu_n(P) \end{aligned}$$

by the definition of Θ_n . Because η is a Poisson process we have

$$\mathbb{P}(\eta \cap P^o = \emptyset) = e^{-\gamma\Phi(P)}.$$

In the following we are interested in the case where $D = \mathfrak{h}_n^{-1}([0, 1]^d \times B \times C)$ with Borel sets $B \subset [0, \infty)$ and $C \subset \mathcal{P}_{n,c,\Phi}$. By (2.7) we obtain in this case

$$\begin{aligned} \mathbb{E}X(\mathfrak{h}_n^{-1}([0, 1]^d \times B \times C)) &= \gamma^n \int_{\mathcal{P}_n} \mathbb{1}(P \in \mathfrak{h}_n^{-1}([0, 1]^d \times B \times C)) e^{-\gamma\Phi(P)} d\mu_n(P) \\ &= \gamma^n \int_C \int_B \int_{[0, 1]^d} d\lambda_d(\mathfrak{c}) e^{-\gamma t} d\lambda_1^{(n-d)}(t) d\mu_{n,c,\Phi}(P). \end{aligned} \quad (2.9)$$

For the first part of Theorem 2.2, observe that by the definition (2.5) of the intensity measure \mathbb{Q} and using (2.9) we have

$$\begin{aligned} \mathbb{P}(f(Z) = n, \mathfrak{s}(Z) \in C) &= \frac{\gamma^{(d)}}{\gamma^{(d)}} \lambda_d([0, 1]^d) \mathbb{Q}(\mathcal{P}_{n, \mathfrak{c}} \cap \mathfrak{s}^{-1}(C)) \\ &= \frac{1}{\gamma^{(d)}} \mathbb{E} X(\{P \in \mathcal{P}_n, \mathfrak{c}(P) \in [0, 1]^d, \mathfrak{s}(P) \in C\}) \\ &= \frac{\gamma^n}{\gamma^{(d)}} \int_C \int_0^\infty \int_{[0, 1]^d} d\lambda_d(\mathbf{c}) e^{-\gamma t} d\lambda_1^{(n-d)}(t) d\mu_{n, \mathfrak{c}, \Phi}(P). \end{aligned}$$

Because the integration with respect to t gives $(n-d)! \gamma^{-(n-d)} \lambda_1^{(n-d)}([0, 1])$ by elementary computations, the right hand side equals

$$\frac{\gamma^d}{\gamma^{(d)}} (n-d)! \mu_n(\{P \in \mathcal{P}_n, \mathfrak{c}(P) \in [0, 1]^d, \Phi(P) < 1, \mathfrak{s}(P) \in C\}).$$

by the definition of μ_n . This proves the first part of the theorem.

Analogously, for the second part we have

$$\begin{aligned} \mathbb{P}(f(Z) = n, \Phi(Z) \in B, \mathfrak{s}(Z) \in C) &= \frac{\gamma^n}{\gamma^{(d)}} \int_C \int_B \int_{[0, 1]^d} d\lambda_d(\mathbf{c}) e^{-\gamma t} d\lambda_1^{(n-d)}(t) d\mu_{n, \mathfrak{c}, \Phi}(P) \\ &= \frac{\gamma^n}{\gamma^{(d)}} (n-d) \mu_{n, \mathfrak{c}, \Phi}(C) \int_B e^{-\gamma t} t^{n-d-1} dt. \end{aligned}$$

Thus, if we condition Z to have n facets, we have that $\mathfrak{s}(Z)$ and $\Phi(Z)$ are independent random variables with distribution

$$\mathbb{P}(\mathfrak{s}(Z) \in C \mid f(Z) = n) = \mu_{n, \mathfrak{c}, \Phi}(C) / \mu_{n, \mathfrak{c}, \Phi}(\mathcal{P}_{n, \mathfrak{c}})$$

and

$$\mathbb{P}(\Phi(Z) \in B \mid f(Z) = n) = \frac{\gamma^{n-d}}{(n-d-1)!} \int_B e^{-\gamma t} t^{n-d-1} dt.$$

3 Polytope approximation

In this section we provide the necessary geometric ingredients for our main results. The tools used here are from convex geometry, in particular approximation of polytopes by polytopes with fewer facets.

3.1 General approximation results for polytopes

The starting point of this subsection is [20] from Reisner, Schütt and Werner. Our goal is to show that if a polytope P has many facets, then a good proportion of them have only a tiny influence. This will be a key ingredient to obtain a recurrence relation between the probabilities $\mathbb{P}(f(Z) = n)$ and $\mathbb{P}(f(Z) = n - 1)$ in Theorem 4.1. More precisely, for $I \subset \mathbb{N}$ and a set of halfspaces $H_i^{\epsilon_i}$, $i \in I$, we define

$$P_I := \cap_{i \in I} H_i^{\epsilon_i}.$$

Throughout the paper we use the notation

$$[n] = \{1, \dots, n\}.$$

For $j \leq n$ we have $P_{[n]} \subset P_{[n] \setminus \{j\}}$. We will measure the distance between $P_{[n]}$ and $P_{[n] \setminus \{i\}}$, both with the Hausdorff distance and the difference of Φ -content. We will show in the crucial Lemma 3.3 that for a subset $J \subset [n]$ of size at least $n/4$ we have good upper bounds of the distances between $P_{[n]}$ and $P_{[n] \setminus \{j\}}$ for $j \in J$.

We first present Lemmata 3.1 and 3.2 which are adaptations of results from [20].

Lemma 3.1. *There exists a constant $c_6 > 1$, independent of d , such that the following holds. For any integer $m > c_6^{(d-1)/2}$ and any $K \subset \mathcal{K}$, there exists a polytope $Q \supset K$ with m facets such that*

$$d_H(K, Q) < c_6 c_\Phi \Phi(K) m^{-\frac{2}{d-1}}.$$

Proof. This is an immediate consequence of Corollary 2.8. of [20] which says that

$$d_H(K, Q) < c_6 R(K) m^{-\frac{2}{d-1}}$$

where $R(K)$ is the circumradius of K , i.e. the radius of the smallest ball containing the convex body K . The lemma follows from

$$V_1(K) \geq V_1([\mathbf{o}, R(K)\mathbf{u}]) = R(K)$$

and (2.4). □

The next lemma shows that if the convex body itself is a polytope $P_{[n]} = \cap_{i=1}^n H_i^-$, then Q can be taken as the intersection P_I of suitable supporting halfspaces of $P_{[n]}$. Its proof is similar to the proof of Lemma 4.3 in [20].

Lemma 3.2. *There exist constants c_7 and $c_8 > 0$, such that the following holds. For any integer $k > c_7$ and any simple polytope $P_{[n]}$ with n facets, there exists a subset $I \subset [n]$ with $|I| \leq k$ such that*

$$d_H(P_{[n]}, P_I) < c_8 c_\Phi \Phi(P_{[n]}) k^{-\frac{2}{d-1}}.$$

Proof. We set $c_7 := d(c_6 + 1)$ where c_6 is the constant of Lemma 3.1. We apply Lemma 3.1 to $P_{[n]}$ and $m = \lfloor k/d \rfloor > c_6^{(d-1)/2}$. We obtain a polytope $Q \supset P_{[n]}$ with $\lfloor k/d \rfloor$ facets and

$$d_H(P_{[n]}, Q) < c_6 c_\Phi \Phi(P_{[n]}) \left\lfloor \frac{k}{d} \right\rfloor^{-\frac{2}{d-1}} < c_8 c_\Phi \Phi(P_{[n]}) k^{-\frac{2}{d-1}}.$$

By eventually shifting and rotating the facets of Q slightly, we can assume that each of the facets of Q meets exactly one vertex of $P_{[n]}$ in its interior. Let I be the set of indices of facets of $P_{[n]}$ with one vertex in a facet of Q . Since $P_{[n]}$ is simple, we have

$$|I| \leq d f(Q) = d \left\lfloor \frac{k}{d} \right\rfloor \leq k.$$

Finally, we observe that $P_{[n]} \subset P_I \subset Q$, which implies $d_H(P_{[n]}, P_I) \leq d_H(P_{[n]}, Q)$. \square

The crucial step is to prove that also the Φ -content between $P_{[n]}$ and $P_{[n] \setminus \{j\}}$ is almost the same for $j \in I$.

Lemma 3.3. *There exist constants c_9 and c_{10} such that the following holds. For any $n > c_9$ and any simple polytope $P_{[n]} = \cap_{i=1}^n H_i^-$ with n facets, there exists a subset $J \subset [n]$ of cardinality at least $n/4$ such that for any $j \in J$ we have*

$$d_H(P_{[n]}, P_{[n] \setminus \{j\}}) < c_{10} c_\Phi \Phi(P_{[n]}) n^{-\frac{2}{d-1}}, \quad (3.1)$$

and

$$\Phi(P_{[n] \setminus \{j\}}) < \exp \left\{ c_{10} c_\Phi n^{-\frac{d+1}{d-1}} \right\} \Phi(P_{[n]}). \quad (3.2)$$

Proof. The first part is just a suitable reformulation of Lemma 3.2. Set $c_9 := 2c_7 + 4$, and put $k = n - 2\lceil n/4 \rceil$ which implies $k \geq c_7$. By Lemma 3.2 there is a set $I \subset \{1, \dots, n\}$ of cardinality k such that

$$d_H(P_{[n]}, P_I) < c_8 c_\Phi \Phi(P_{[n]}) k^{-\frac{2}{d-1}} \leq (4d)^{-1} c_{10} c_\Phi \Phi(P_{[n]}) n^{-\frac{2}{d-1}}$$

for a suitable constant c_{10} by the definition of k . Hence for any $j \notin I$,

$$d_H(P_{[n]}, P_{[n] \setminus \{j\}}) \leq d_H(P_{[n]}, P_I) < (4d)^{-1} c_{10} c_\Phi \Phi(P_{[n]}) n^{-\frac{2}{d-1}} \quad (3.3)$$

which gives (3.1). It remains to show that, for at least half of the j not in I , equation (3.2) holds as well. Set $\delta = (4d)^{-1} c_{10} c_\Phi \Phi(P_{[n]}) n^{-2/(d-1)}$ and

$$U_j = \text{cl}\{\mathbf{u} \in \mathbb{S}^{d-1} : h(P_{[n] \setminus \{j\}}, \mathbf{u}) \neq h(P_{[n]}, \mathbf{u})\}.$$

Equation (3.3) implies that, for any $j \notin I$ and $\mathbf{u} \in \mathbb{S}^{d-1}$, we have

$$0 \leq h(P_{[n] \setminus \{j\}}, \mathbf{u}) - h(P_{[n]}, \mathbf{u}) \leq \delta \mathbf{1}(\mathbf{u} \in U_j).$$

Therefore,

$$\begin{aligned}\Phi(P_{[n]\setminus\{j\}}) - \Phi(P_{[n]}) &= \int_{\mathbb{S}^{d-1}} h(P_{[n]\setminus\{j\}}, \mathbf{u}) - h(P_{[n]}, \mathbf{u}) \, d\varphi(\mathbf{u}) \\ &< \int_{U_j} \delta \, d\varphi(\mathbf{u}) = \delta\varphi(U_j).\end{aligned}\tag{3.4}$$

We need to estimate the φ -measure of the set U_j . Denote by $\mathbf{v}_1, \dots, \mathbf{v}_m$ the vertices of the polytope P . Since the polytope is simple, each vertex is the intersection of precisely d hyperplanes. Denote by $N(\mathbf{v}_l)$ the unit vectors in the normal cone of P at \mathbf{v}_l , i.e.

$$N(\mathbf{v}_l) = \{\mathbf{u} \in \mathbb{S}^{d-1} : h(P_{[n]}, \mathbf{u}) = \mathbf{v}_l \cdot \mathbf{u}\}.$$

The essential observation is that

$$U_j = \bigcup_{\mathbf{v}_l \in H_j} N(\mathbf{v}_l).$$

Observe that the sets $N(\mathbf{v}_l)$ have pairwise disjoint interiors and cover \mathbb{S}^{d-1} . Thus for almost all $\mathbf{u} \in \mathbb{S}^{d-1}$ we have

$$\begin{aligned}\sum_{j=1}^n \mathbb{1}(\mathbf{u} \in U_j) &= \sum_{j=1}^n \sum_{l=1}^m \mathbb{1}(\mathbf{v}_l \in H_j) \mathbb{1}(\mathbf{u} \in N(\mathbf{v}_l)) \\ &= \underbrace{\sum_{l=1}^m \mathbb{1}(\mathbf{u} \in N(\mathbf{v}_l))}_{=1} \underbrace{\sum_{j=1}^n \mathbb{1}(\mathbf{v}_l \in H_j)}_{=d} = d\end{aligned}$$

This yields $\sum_{j=1}^n \varphi(U_j) = d$ and in particular

$$\sum_{j \notin I} \varphi(U_j) \leq d.$$

This implies that, for at least half of the $j \notin I$, we have

$$\varphi(U_j) \leq d \left(\frac{n-k}{2} \right)^{-1} = d \left\lceil \frac{n}{4} \right\rceil^{-1} \leq 4dn^{-1}.$$

Otherwise we would have at least half of the $j \notin I$ with the reverse inequality and, because $|I| = k = n - 2\lceil n/4 \rceil$, that would imply

$$d \geq \sum_{j \notin I} \varphi(U_j) > \frac{1}{2}(n-k) \frac{2d}{n-k} = d.$$

Combined with equation (3.4), it shows that there exists a set $J \subset \{1, \dots, n\} \setminus I$ of cardinality $(n-k)/2 = \lceil n/4 \rceil$ such that, for any $j \in J$, we have

$$\Phi(P_{[n]\setminus\{j\}}) - \Phi(P_{[n]}) < 4d\delta n^{-1} = c_{10}c_\Phi n^{-\frac{d+1}{d-1}} \Phi(P_{[n]}).$$

This implies equation (3.2). □

3.2 Approximation with elongation condition

The starting point of the following considerations is Theorem 1.1 of [1]. For $1 \leq i < j \leq d$ and $\epsilon > 0$, we say that a convex body K is $(\epsilon: i, j)$ -elongated when $V_j(K)^{1/j} V_i(K)^{-1/i} < \epsilon$. When a convex body, or more specifically a polytope, is sufficiently elongated, the approximation results of the previous subsection can be improved.

Lemma 3.4. *Assume that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. There exist positive constants c_{11} and c_{12} , both depending on i, j and d , such that the following holds. For any $\epsilon > 0$, any integer $k \geq \lfloor c_{11} \epsilon^{-(d-2)} \rfloor$ and any simple polytope $P_{[n]} = \cap_{i=1}^n H_i^{\epsilon_i} \in \mathcal{P}_n$ with $n \geq k$ facets and $V_j(P_{[n]})^{1/j} V_i(P_{[n]})^{-1/i} < \epsilon$, there exists a subset $J \subset [n]$ with $|J| \leq k$, such that*

$$d_H(P_{[n]}, P_J) < c_{12} \epsilon^{\frac{1}{2d}} V_1(P_{[n]}) k^{-\frac{2}{d-1}}.$$

Proof. This is a useful application of a recent result by Bonnet [1]. Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. Then there exist constants $c_{i,j}$ and $n_{i,j}$ (both depending on d), such that the following holds. For any $\epsilon > 0$, any $m \geq n_{i,j} \epsilon^{-(d-2)}$, and any convex body K with

$$\frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} < \epsilon,$$

there exists a polytope $Q \supset K$ with at most m facets satisfying

$$d_H(K, Q) < c_{i,j} \epsilon^{\frac{1}{2d}} V_1(K) m^{-\frac{2}{d-1}}.$$

Assume that $P_{[n]}$ is a simple polytope with isoperimetric ratio $V_j(P_{[n]})^{1/j} V_i(P_{[n]})^{-1/i} < \epsilon$ and $f(P_{[n]}) = n > k \geq dm$ facets with $m = \lfloor k/d \rfloor > n_{i,j} \epsilon^{-(d-2)}$. Then there exists a polytope $Q \supset P_{[n]}$ with $m+1$ facets and

$$d_H(P_{[n]}, Q) < c_{i,j} \epsilon^{\frac{1}{2d}} V_1(P_{[n]}) (m+1)^{-\frac{2}{d-1}} \leq d^{\frac{2}{d-1}} c_{i,j} \epsilon^{\frac{1}{2d}} V_1(P_{[n]}) k^{-\frac{2}{d-1}}.$$

We can assume that each of the facets of Q meets exactly one vertex of $P_{[n]}$ in its interior. Let J be the set of indices of facets of $P_{[n]}$ with one vertex in a facet of Q . Since $P_{[n]}$ is simple, we have

$$|J| \leq d f(Q) \leq k.$$

And $P_J \subset Q$ implies

$$d_H(P_{[n]}, P_J) \leq d_H(P_{[n]}, Q).$$

□

In the following lemma we prove the uniform continuity of the isoperimetric ratio. To our surprise we could not find any results in this direction, this seems to be an open problem. We state the partial solution to this problem which we need for our purposes.

Lemma 3.5. *Let $1 \leq i < j \leq d$. There exists a constant c_{13} such that for any $\delta \in (0, 1)$ and for any $K, L \in \mathcal{K}$ with $K \subset L$ and $d_H(K, L) < \delta V_1(K)$, we have*

$$\frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} < \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{13} \delta^{\frac{j-i}{ij(j-1)}}.$$

Proof. A first easy bound is obtained using $V_i(L)^{j-1} \geq c_{ij} V_1(L)^{j-i} V_j(L)^{i-1}$ which is a consequence of the Alexandrov-Fenchel inequality, see [21], p.401, (7.66) therein.

$$\frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} \leq c_{14} \left(\frac{V_i(L)^{\frac{1}{i}}}{V_1(L)} \right)^{\frac{j-i}{j(i-1)}} < \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{14} \left(\frac{V_i(L)^{\frac{1}{i}}}{V_1(L)} \right)^{\frac{j-i}{j(i-1)}} \quad (3.5)$$

A more precise bound uses Steiner's formula. Due to the isoperimetric inequality (2.2), $V_i(K)^{1/i} \leq c_{1,i} V_1(K)$ with $c_{1,i} := V_i(B^d)^{1/i} V_1(B^d)$. Since

$$d_H(K, L) < \delta V_1(K),$$

we have that $L \subset K + \delta V_1(K) B^d$. The monotonicity of the intrinsic volumes and Steiner's formula (2.1) shows for $\delta \leq 1$

$$\begin{aligned} V_j(L) &< V_j(K + \delta V_1(K) B^d) \\ &\leq V_j(K) + \sum_{i=1}^j \binom{d-j+i}{i} \frac{\kappa_{d-j+i}}{\kappa_{d-j}} c_{1,j-i}^{j-i} V_1(K)^{j-i} (\delta V_1(K))^i \\ &\leq V_j(K) + \delta V_1(K)^j \sum_{i=1}^j \binom{d-j+i}{i} \frac{\kappa_{d-j+i}}{\kappa_{d-j}} c_{1,j-i}^{j-i} \\ &\leq V_j(K) + c_{15} \delta V_1(L)^j. \end{aligned}$$

Because $a + b \leq (a^{\frac{1}{j}} + b^{\frac{1}{j}})^j$ for $a, b > 0$, and because of the monotonicity of the intrinsic volumes this yields

$$\frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} \leq \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{15}^{\frac{1}{j}} \delta^{\frac{1}{j}} \frac{V_1(L)}{V_i(L)^{\frac{1}{i}}}. \quad (3.6)$$

Note that $\min\{x, x^{-(j-i)/(j(i-1))}\} \leq 1$ for all $x > 0$. We define $c_{13} = \max\{c_{14}, c_{15}^{1/j}\}$ and combine (3.5) and (3.6).

$$\begin{aligned} \frac{V_j(L)^{\frac{1}{j}}}{V_i(L)^{\frac{1}{i}}} &\leq \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + \delta^{\frac{j-i}{ij(j-1)}} \min \left\{ c_{15}^{\frac{1}{j}} \delta^{\frac{i-1}{i(j-1)}} \frac{V_1(L)}{V_i(L)^{\frac{1}{i}}}, c_{14} \left(\delta^{\frac{i-1}{i(j-1)}} \frac{V_1(L)}{V_i(L)^{\frac{1}{i}}} \right)^{-\frac{j-i}{j(i-1)}} \right\} \\ &\leq \frac{V_j(K)^{\frac{1}{j}}}{V_i(K)^{\frac{1}{i}}} + c_{13} \delta^{\frac{j-i}{ij(j-1)}} \end{aligned}$$

□

Recall that we use the notation $P_I = \cap_{i \in I} H_i^{\epsilon_i}$, for any set of integers I . For integers $k \leq n$ and a permutation $\sigma \in \mathfrak{S}_n$ we write $\sigma[k] = \{\sigma(i) : i \in [k]\}$. In particular $P_{\sigma[k]} = \cap_{i \in I} H_{\sigma(i)}^{\epsilon_{\sigma(i)}}$. We call hyperplanes H_i in generic position, if the intersection of any $d+2$ of them is empty. The constants c_{11} and c_{12} have been defined in Lemma 3.4.

Lemma 3.6. *There is a constant c_{16} depending on φ such that for all integers $i < j \leq \lceil (d-1)/2 \rceil$ and any $\epsilon < c_{11}^{2/(d-1)} c_{12}^{-1} (c_\Phi)^{-1}$ the following holds. For any polytope $P_{[n]} \in \mathcal{P}_n$ with $n > m = \lfloor c_{11} \epsilon^{-(d-2)} \rfloor$ facets in generic position and $V_j(P_{[n]})^{1/j} V_i(P_{[n]})^{-1/i} < \epsilon$ there exist at least $2^{-n}(n-2m)!$ permutations $\sigma \in \mathfrak{S}_n$ such that*

- (1) $d_H(P_{\sigma[k]}, P_{\sigma[k-1]}) < c_{16} c_\Phi \epsilon^{\frac{1}{2d^4}} \Phi(P_{\sigma[m]}) k^{-\frac{2}{d-1}}$ for all $k = 2m+1, \dots, n$,
- (2) $\|\mathbf{c}(P_{\sigma[n]}) - \mathbf{c}(P_{\sigma[m]})\| < \Phi(P_{\sigma[n]})$, and
- (3) $\Phi(P_{\sigma[m]}) < 2\Phi(P_{\sigma[n]})$.

Proof. We set

$$m = \lfloor c_{11} \epsilon^{-(d-2)} \rfloor.$$

By Lemma 3.4 there exists a subset $I \subset [n]$ with $|I| = m$, such that for all subsets J with $I \subset J \subset [n]$ we have

$$d_H(P_{[n]}, P_J) < c_{12} \epsilon^{\frac{1}{2d}} V_1(P_{[n]}) m^{-\frac{2}{d-1}} < c_{11}^{-\frac{2}{d-1}} c_{12} \epsilon V_1(P_{[n]}).$$

By Lemma 3.5 this implies for all such sets J that

$$\frac{V_j(P_J)^{\frac{1}{j}}}{V_i(P_J)^{\frac{1}{i}}} < \epsilon + c_{13} (c_{11}^{-\frac{2}{d-1}} c_{12} \epsilon)^{\frac{j-i}{ij(j-1)}} < c_{17} \epsilon^{\frac{1}{d^3}}. \quad (3.7)$$

We denote by $S(P_{[n]}) \subset \mathfrak{S}_n$ the set of those permutations σ such that

- (a) $\sigma[m] = I$, and
- (b) $d_H(P_{\sigma[k]}, P_{\sigma[k-1]}) < 2^{\frac{2}{d-1}} c_{12} c_{17}^{\frac{1}{2d}} \epsilon^{\frac{1}{2d^4}} V_1(P_{\sigma[k]}) k^{-\frac{2}{d-1}}$ for all $k = n, \dots, 2m+1$.

To estimate $|S(P_{[n]})|$ note first that there are $m!$ possibilities such that $\sigma[m] = I$. Second, assume that $\sigma(n), \dots, \sigma(k+1) \in [n] \setminus I$ are already chosen satisfying Condition (b). Then by (3.7) and by Lemma 3.4 applied to the polytope $P' = P_{\sigma[k]}$, the integer $k' = \frac{k}{2} \geq m$ and $\epsilon' = c_{17} \epsilon^{\frac{1}{d^3}}$, there is a set $J_k \subset \sigma[k]$ of size $|J_k| \leq k/2$ such that

$$d_H(P_{\sigma[k]}, P_{J_k}) < c_{12} c_{17}^{\frac{1}{2d}} \epsilon^{\frac{1}{2d^4}} V_1(P_{\sigma[k]}) \left(\frac{k}{2}\right)^{-\frac{2}{d-1}}.$$

If we choose $\sigma(k) \notin J_k$, Condition (b) is thus satisfied. Because we need in addition $\sigma(k) \notin I$ there are at least $k/2 - m$ possibilities to choose $\sigma(k)$, and thus to determine

$\sigma[k-1]$. Continuing until $k = 2m + 1$ gives at least $\prod_{2m+1}^n (k/2 - m)$ possibilities to choose $\sigma(n), \dots, \sigma(2m + 1)$. We obtain

$$|S(P_{[n]})| \geq m! \prod_{k=2m+1}^n \left(\frac{k}{2} - m \right) = m! 2^{-n+2m} (n - 2m)! > 2^{-n} (n - 2m)!$$

Using (2.4), we observe that Condition (1) of our lemma is satisfied by choosing $c_{16} = 2^{\frac{2}{d-1}} c_{12} c_{17}^{\frac{1}{2d}}$ in Condition (b). Condition (2) follows from the 1-Lipschitz property of \mathbf{c} and

$$d_H(P_{[n]}, P_{\sigma[m]}) < c_{11}^{-\frac{2}{d-1}} c_{12} c_{\Phi} \Phi(P_{[n]}) \epsilon < \Phi(P_{[n]}),$$

if $c_{11}^{-\frac{2}{d-1}} c_{12} c_{\Phi} \epsilon < 1$. Condition (3) follows from this and the fact that

$$\Phi(P_{\sigma[m]}) < \Phi(P_{\sigma[n]} + \Phi(P_{\sigma[n]}) B^d) < 2\Phi(P_{\sigma[n]}).$$

□

4 Cells with many facets

The aim of this section is to show how Theorem 2.2, combined with the geometric arguments developed in Section 3, implies our main results, i.e. Theorems 1.1, 1.2 and 1.3. To do so, we start by presenting the three intermediary results that will play a key role in the proofs of these theorems.

By seeing a polytope with n facets as a polytope with $(n - 1)$ facets cut ‘a little bit’ by one halfspace, we obtain the following recurrence relation.

Theorem 4.1. *There exist constants c_{18} and c_{19} , independent of φ , such that for $n > (c_{18} c_{\Phi})^{d/2}$,*

$$\mathbb{P}(f(Z) = n) \leq c_{19} c_{\Phi} n^{-\frac{2}{d-1}} \mathbb{P}(f(Z) = n - 1)$$

and $c_{19} c_{\Phi} n^{-2/(d-1)} < 1$, where c_{Φ} is defined in (2.4).

The next theorem provides an upper-bound for the probability of the same event $\{f(Z) = n\}$ intersected with the event of being $(\epsilon: i, j)$ -elongated. Let c_{11} , c_{12} , and c_{Φ} be the constants used in Lemma 3.6.

Theorem 4.2. *Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. There exist constants c_{20} , c_{21} and c_{22} , such that for any $\epsilon < c_{11}^{2/(d-1)} c_{12}^{-1} (c_{\Phi})^{-1}$ we have*

$$\mathbb{P} \left(f(Z) = n, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \right) < \frac{\gamma^d}{\gamma^{(d)}} e^{c_{21} \epsilon^{-2(d-1)}} (c_{22} \epsilon^{\frac{1}{2d^4}})^n n^{-\frac{2n}{d-1}}$$

for $n > \lfloor c_{11} \epsilon^{-(d-2)} \rfloor^2$.

As we will see, the bound in Theorem 4.2, i.e. when we add the condition that the isoperimetric ratio is small, is close to the one that we will get when iterating Theorem 4.1 but with a constant in front of $n^{-\frac{2n}{d-1}}$ which is arbitrarily small.

The last theorem deals with the lower-bound for $\mathbb{P}(f(Z) = n)$. This requires an extra-condition on the directional distribution φ . Recall that we call φ *well spread* if there exists a cap on the unit sphere where φ is bounded from below by a multiple of the surface area measure. We denote by $\mathcal{H}^{d-1}(\cdot)$ the $(d-1)$ -dimensional Hausdorff measure, the surface area. We will prove a slightly more precise form of Theorem 1.2

Theorem 4.3. *There exist constants $c_{23} > 0$, $c_{24} \in \mathbb{N}$, independent of φ , such that the following holds. Assume that φ is well spread. In particular assume that there exists a cap $C \subset \mathbb{S}^{d-1}$ of radius $r \in (0, 1)$ and a constant c_{25} with $\varphi(\cdot) > c_{25} \mathcal{H}^{d-1}(\cdot)$ on C . Then, for $n \geq c_{24}$, we have*

$$\mathbb{P}(f(Z) = n) > \frac{\gamma^d}{\gamma^{(d)}} (c_{25} c_{23} r^{d+2})^n n^{-\frac{2n}{d-1}}.$$

In the next subsection, we show how to deduce in a very small number of steps our main results from the three theorems above. The rest of Section 4 is devoted to the proof of Theorems 4.1, 4.2 and 4.3.

4.1 Deducing Theorems 1.1, 1.2 and 1.3 from Theorems 4.1, 4.3 and 4.2

Set $n_0 := \lceil (c_{18} c_\Phi)^{d/2} \rceil$ where c_{18} is given by Theorem 4.1. Iterating Theorem 4.1, gives us that for any $n \geq n_0$,

$$\mathbb{P}(f(Z) = n) \leq (c_{19} c_\Phi)^{n-n_0} \left(\frac{n!}{n_0!} \right)^{-\frac{2}{d-1}}.$$

Now Stirling's approximation $n! > n^n e^{-n}$ implies for any $n \geq n_0$,

$$\mathbb{P}(f(Z) = n) < (c_{19} c_\Phi)^{-n_0} (n_0!)^{\frac{2}{d-1}} (e^{\frac{2}{d-1}} c_{19} c_\Phi)^n n^{-\frac{2n}{d-1}}$$

which implies Theorem 1.1.

Taking $c_2 = \left(\min(1, \frac{\gamma^d}{\gamma^{(d)}})^{\frac{1}{c_{24}}} \right) c_{25} c_{23} r^{d+2}$ where c_{24} , c_{25} , c_{23} and r are given by Theorem 4.3, we obtain Theorem 1.2.

Taking $c_{26} = c_{11}^{2/(d-1)} c_{12}^{-1} (c_\Phi)^{-1}$, $c_{27} = \frac{c_{22} \epsilon^{\frac{1}{2d^4}}}{c_{25} c_{23} r^{d+2}}$ and $c_{28}^{(\epsilon)} = e^{c_{21} \epsilon^{-2(d-1)}}$, we deduce from Theorem 4.3 and Theorem 4.2, when φ is well spread, that

$$\mathbb{P} \left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \mid f(Z) = n \right) \leq c_{28}^{(\epsilon)} (c_{27} \epsilon^{\frac{1}{2d^4}})^n$$

for any $\epsilon < c_{26}$ and $n \geq \max(c_{24}, \lfloor c_{11} \epsilon^{-(d-2)} \rfloor^2)$. Now, choose ϵ such that $c_{27} \epsilon^{\frac{1}{2d^4}} = \delta/2$. Theorem 1.3 follows from the fact that $c_{28}^{(\epsilon)} (\frac{\delta}{2})^n < \delta^n$, for $n > \ln(c_{28}^{(\epsilon)}) / \ln(2)$.

4.2 Proof of Theorem 4.1

We first need to state the following elementary but useful lemma. We denote by \mathfrak{S}_n the set of permutations of $[n]$. For $\mathbf{x} = (x_1, \dots, x_n)$ and $\sigma \in \mathfrak{S}_n$, we write $\mathbf{x}_\sigma := (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. It is clear that the following holds.

Lemma 4.4. *Let (X, Σ, ψ) be a measured space, $m, n > 0$ be integers, $f : X^n \rightarrow [0, \infty)$ be a measurable function and $S, T \subset X^n$ measurable sets. Assume that*

- *f is symmetric: for any $\sigma \in \mathfrak{S}_n$ and any $\mathbf{x} \in X^n$, we have $f(\mathbf{x}_\sigma) = f(\mathbf{x})$;*
- *S is symmetric: for any $\sigma \in \mathfrak{S}_n$, and any $\mathbf{x} \in X^n$ we have $\mathbb{1}(\mathbf{x}_\sigma \in S) = \mathbb{1}(\mathbf{x} \in S)$;*
- *for any $\mathbf{x} \in S$, there exist at least p permutations $\sigma \in \mathfrak{S}_n$ such that $\mathbf{x}_\sigma \in T$.*

Then

$$\frac{p}{n!} \int_{X^n} \mathbb{1}(\mathbf{x} \in S) f(\mathbf{x}) d\psi^n(\mathbf{x}) \leq \int_{X^n} \mathbb{1}(\mathbf{x} \in T) f(\mathbf{x}) d\psi^n(\mathbf{x}).$$

The next lemma deals with the measure of those polytopes $P_{[n]}$ which are close to $P_{[n-1]}$ in the Hausdorff distance.

Lemma 4.5. *For any $\varepsilon > 0$ and any measurable function $f : \tilde{\mathcal{H}}^{n-1} \rightarrow (0, \infty)$, it holds that*

$$\begin{aligned} \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(d_H(P_{[n]}, P_{[n-1]}) < \varepsilon) f(H_1^{\varepsilon_1}, \dots, H_{n-1}^{\varepsilon_{n-1}}) d\tilde{\mu}^n(\mathbf{H}^\varepsilon) \\ < \varepsilon \int_{\tilde{\mathcal{H}}^{n-1}} \mathbb{1}(P_{[n-1]} \in \mathcal{P}_{n-1}) f(\mathbf{H}^\varepsilon) d\tilde{\mu}^{n-1}(\mathbf{H}^\varepsilon). \end{aligned} \quad (4.1)$$

Proof. In this particular proof, we use the following representation of half-spaces: for any $\mathbf{u} \in \mathbb{S}^{d-1}$ and $t \in \mathbb{R}$, we denote by $\tilde{H}(\mathbf{u}, t) = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{u} \rangle \leq t\}$ so that

$$\tilde{\mu}(\cdot) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathbb{1}(\tilde{H}(\mathbf{u}, t) \in \cdot) dt d\varphi(\mathbf{u}).$$

For any $K \in \mathcal{K}$, we now proceed with the following calculation:

$$\begin{aligned} & \int_{\tilde{\mathcal{H}}} \mathbb{1}(K \cap H_n \neq \emptyset) \mathbb{1}(d_H(K, K \cap H_n^{\varepsilon_n}) < \varepsilon) d\tilde{\mu}(H_n^{\varepsilon_n}) \\ &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathbb{1}(K \cap \tilde{H}(\mathbf{u}, t) \neq \emptyset) \mathbb{1}(d_H(K, K \cap \tilde{H}(\mathbf{u}, t)) < \varepsilon) dt d\varphi(\mathbf{u}) \\ &\leq \int_{\mathbb{S}^{d-1}} \int_{h(K, \mathbf{u}) - \varepsilon}^{h(K, \mathbf{u})} dt d\varphi(\mathbf{u}) = \varepsilon. \end{aligned} \quad (4.2)$$

Let us fix now $\mathbf{H}^\epsilon \in \tilde{\mathcal{H}}^{n-1}$. We observe that for every $H_n^\epsilon \in \tilde{\mathcal{H}}$,

$$\begin{aligned} & \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(d_H(P_{[n]}, P_{[n-1]}) < \varepsilon) \\ & \leq \mathbb{1}(P_{[n-1]} \in \mathcal{P}_{n-1}) \mathbb{1}(P_{[n-1]} \cap H_n \neq \emptyset) \mathbb{1}(d_H(P_{[n-1]}, P_{[n-1]} \cap H_n^\epsilon) < \varepsilon). \end{aligned} \quad (4.3)$$

Integrating (4.3) over $H_n^\epsilon \in \tilde{\mathcal{H}}$ and combining it with (4.2) applied to $K = P_{[n-1]}$, we obtain

$$\int_{\tilde{\mathcal{H}}} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(d_H(P_{[n]}, P_{[n-1]}) < \varepsilon) d\tilde{\mu}(H_n^\epsilon) \leq \varepsilon \mathbb{1}(P_{[n-1]} \in \mathcal{P}_{n-1}).$$

We conclude by multiplying the previous inequality by $f(H_1^{\epsilon_1}, \dots, H_{n-1}^{\epsilon_{n-1}})$ and integrating it with respect to $(H_1^{\epsilon_1}, \dots, H_{n-1}^{\epsilon_{n-1}}) \in d\tilde{\mu}^{n-1}(\mathbf{H}^\epsilon)$. \square

We are now in the position to prove Theorem 4.1. Set $\alpha = c_{10}c_\Phi n^{-2/(d-1)}$, and set

$$I_n = \frac{\gamma^{(d)}}{\gamma^d} \frac{n!}{(n-d)!} \mathbb{P}(f(Z) = n).$$

By (2.6) and (2.8), we have

$$I_n = \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(\mathbf{c}(P_{[n]}) \in [0, 1]^d) \mathbb{1}(\Phi(P_{[n]}) < 1) d\tilde{\mu}^n(\mathbf{H}^\epsilon),$$

where $P_{[n]} = \cap_{i=1}^n H_i^{\epsilon_i}$. We want to use now Lemma 3.3 which, roughly speaking, tells us that the variable H_n has a ‘small influence’. Set

$$S = \{(\mathbf{H}, \epsilon) \in \tilde{\mathcal{H}}^n : \cap_{i=1}^n H_i^{\epsilon_i} \in \mathcal{P}_n \text{ is a simple polytope}\},$$

and

$$T = \{(\mathbf{H}, \epsilon) \in S : d_H(P_{[n]}, P_{[n-1]}) < \alpha \Phi(P_{[n]}), \Phi(P_{[n-1]}) < \exp\{\alpha n^{-1}\} \Phi(P_{[n]})\}.$$

Lemma 3.3 tells us that, for any $(\mathbf{H}, \epsilon) \in S$, there exists at least $n!/4$ permutations $\sigma \in \mathfrak{S}_n$ such that $(\mathbf{H}, \epsilon)_\sigma \in T$. Hence, Lemma 4.4 and the Lipschitz continuity of \mathbf{c} imply

$$\begin{aligned} \frac{I_n}{4} & \leq \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(\mathbf{c}(P_{[n]}) \in [0, 1]^d) \\ & \quad \mathbb{1}(d_H(P_{[n]}, P_{[n-1]}) < \alpha) \mathbb{1}(\Phi(P_{[n-1]}) < \exp\{\alpha n^{-1}\}) d\tilde{\mu}^n(\mathbf{H}^\epsilon) \\ & \leq \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(\mathbf{c}(P_{[n-1]}) \in [-\alpha, 1 + \alpha]^d) \\ & \quad \mathbb{1}(d_H(P_{[n]}, P_{[n-1]}) < \alpha) \mathbb{1}(\Phi(P_{[n-1]}) < \exp\{\alpha n^{-1}\}) d\tilde{\mu}^n(\mathbf{H}^\epsilon). \end{aligned}$$

Now, using consecutively Lemma 4.5 and Lemma 2.1 applied to an $(n - 1)$ -fold integral, we obtain that

$$\begin{aligned}
\frac{I_n}{4} &\leq \alpha \int_{\tilde{\mathcal{H}}^{n-1}} \mathbb{1}(P_{[n-1]} \in \mathcal{P}_{n-1}) \mathbb{1}(\mathfrak{c}(P_{[n-1]}) \in [-\alpha, 1 + \alpha]^d) \\
&\quad \mathbb{1}(\Phi(P_{[n-1]}) < \exp\{\alpha n^{-1}\}) d\tilde{\mu}^{n-1}(\mathbf{H}^\epsilon) \\
&\leq \alpha(1 + 2\alpha)^d \exp\{\alpha(n - 1 - d)n^{-1}\} \\
&\quad \cdot \int_{\tilde{\mathcal{H}}^{n-1}} \mathbb{1}(P_{[n-1]} \in \mathcal{P}_{n-1}) \mathbb{1}(\mathfrak{c}(P_{[n-1]}) \in [0, 1]^d) \mathbb{1}(\Phi(P_{[n-1]}) < 1) d\tilde{\mu}^{n-1}(\mathbf{H}^\epsilon) \\
&\leq \alpha(1 + 2\alpha)^d \exp\left\{\alpha\left(1 - \frac{d+1}{n}\right)\right\} I_{n-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(f(Z) = n) &\leq 4\alpha(1 + 2\alpha)^d \exp\left\{\alpha\left(1 - \frac{d+1}{n}\right)\right\} \frac{(n - d)}{n} \mathbb{P}(f(Z) = n - 1) \\
&\leq 4\alpha \exp\left\{2d\alpha + \alpha\left(1 - \frac{d+1}{n}\right)\right\} \mathbb{P}(f(Z) = n - 1).
\end{aligned}$$

This proves that for $n > c_9$, we have

$$\mathbb{P}(f(Z) = n) < 4\alpha \exp\{3d\alpha\} \mathbb{P}(f(Z) = n - 1).$$

Set $c_{19} := 4e^{3d}c_{10}$ and recall that $\alpha = c_{10}c_\Phi n^{-2/(d-1)}$. Theorem 4.1 follows for $c_{10}c_\Phi n^{-2/(d-1)} < 1$, since in that case $4\alpha e^{3d\alpha} \leq 4\alpha e^{3d} = c_{19}c_\Phi n^{-2/(d-1)}$.

4.3 Proof of Theorem 4.2

We will proceed in a similar way as in the proof of Theorem 4.1 with one main difference: in order to take into account the elongation condition, we will use Lemma 3.6 instead of Lemma 3.3. In particular, Lemma 3.6 does not guarantee that there is a permutation σ such that the condition on the isoperimetric ratio $\frac{V_j(P_{[n]})^{\frac{1}{j}}}{V_i(P_{[n]})^{\frac{1}{i}}} < \epsilon$ is satisfied by $P_{\sigma[n-1]}$ so there is no possibility to apply it more than once. This explains why we have directly a general upper bound but not a recurrence relation similar as the one of Theorem 4.1.

Let c_{11} , c_{12} , c_{16} and c_Φ be the constants used in Lemma 3.6, assume $\epsilon < c_{11}^{2/(d-1)} c_{12}^{-1} (c_\Phi)^{-1}$, and set $m = \lfloor c_{11} \epsilon^{-(d-2)} \rfloor$ and

$$\delta = c_{16} c_\Phi \epsilon^{\frac{1}{2d^4}}.$$

Set

$$I_n := \frac{\gamma^{(d)}}{\gamma^d} \frac{n!}{(n-d)!} \mathbb{P} \left(f(Z) = n, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \right).$$

By (2.8), we have

$$I_n = \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(\mathbf{c}(P_{[n]}) \in [0, 1]^d) \mathbb{1}(\Phi(P_{[n]}) < 1) \mathbb{1} \left(\frac{V_j(P_{[n]})^{\frac{1}{j}}}{V_i(P_{[n]})^{\frac{1}{i}}} < \epsilon \right) d\tilde{\mu}^n(\mathbf{H}^\epsilon),$$

where $P_{[n]} = \cap_{i=1}^n H_i^{\epsilon_i}$. Roughly speaking, we will now use Lemmata 4.4 and 3.6 to order the halfspaces such that integrating step by step, starting by $H_n^{\epsilon_n}$, the integrals can be well bounded. Set

$$S = \left\{ (\mathbf{H}, \epsilon) \in \tilde{\mathcal{H}}^n : P_{[n]} \in \mathcal{P}_n \text{ with facets in generic position, } \frac{V_j(P_{[n]})^{\frac{1}{j}}}{V_i(P_{[n]})^{\frac{1}{i}}} < \epsilon \right\},$$

and

$$T = \left\{ (\mathbf{H}, \epsilon) \in \tilde{\mathcal{H}}^n : P_{[n]} \in \mathcal{P}_n, \|\mathbf{c}(P_{[n]}) - \mathbf{c}(P_{[m]})\| < \Phi(P_{[n]}), \Phi(P_{[m]}) < 2\Phi(P_{[n]}), \right. \\ \left. d_H(P_{[k]}, P_{[k-1]}) < \delta\Phi(P_{[m]})k^{-\frac{2}{d-1}} \text{ for } 2m < k \leq n \right\}.$$

Lemma 3.6 tells us that, for any $(\mathbf{H}, \epsilon) \in S$, there exist at least $2^{-n}(n-2m)!$ permutations $\sigma \in \mathfrak{S}_n$ such that $(\mathbf{H}, \epsilon)_\sigma \in T$. Hence, Lemma 4.4 implies

$$\begin{aligned} & \frac{2^{-n}(n-2m)!}{n!} I_n \\ & \leq \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(\mathbf{c}(P_{[n]}) \in [0, 1]^d) \mathbb{1}(\Phi(P_{[n]}) < 1) \\ & \quad \mathbb{1}(\|\mathbf{c}(P_{[n]}) - \mathbf{c}(P_{[m]})\| < \Phi(P_{[n]})) \mathbb{1}(\Phi(P_{[m]}) < 2\Phi(P_{[n]})) \\ & \quad \mathbb{1}(d_H(P_{[k]}, P_{[k-1]}) < \delta\Phi(P_{[m]})k^{-\frac{2}{d-1}} \text{ for } 2m < k \leq n) d\tilde{\mu}^n(\mathbf{H}^\epsilon) \\ & \leq \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(P_{[n]} \in \mathcal{P}_n) \mathbb{1}(\mathbf{c}(P_{[m]}) \in [-1, 2]^d) \mathbb{1}(\Phi(P_{[m]}) < 2) \\ & \quad \mathbb{1}(d_H(P_{[k]}, P_{[k-1]}) \in (0, 2\delta k^{-\frac{2}{d-1}}) \text{ for } 2m < k \leq n) d\tilde{\mu}^n(\mathbf{H}^\epsilon). \end{aligned}$$

Now, using $n - 2m$ times (4.1), we have

$$\frac{2^{-n}(n-2m)!}{n!} I_n < (2\delta)^{n-2m} \left(\frac{n!}{(2m)!} \right)^{-\frac{2}{d-1}} c_{20},$$

where $c_{20} = c_{20}(m, d, \varphi)$ is defined by

$$c_{20} := \int_{\tilde{\mathcal{H}}^{2m}} \mathbb{1}(P_{[2m]} \in \mathcal{P}_{2m}) \mathbb{1}(\mathbf{c}(P_{[m]}) \in [-1, 2]^d) \mathbb{1}(\Phi(P_{[m]}) < 2) d\tilde{\mu}^{2m}(\mathbf{H}^\epsilon).$$

This implies for $n > m^2$

$$\begin{aligned} \mathbb{P} \left(f(Z) = n, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \right) &< \frac{\gamma^d}{\gamma^{(d)}} c_{20} (2\delta)^{-2m} ((2m)!)^{\frac{2}{d-1}} n^{2m-d} (4\delta)^n (n!)^{-\frac{2}{d-1}} \\ &\leq \frac{\gamma^d}{\gamma^{(d)}} c_{20} g(\epsilon) f(\epsilon)^n n^{-\frac{2n}{d-1}} \end{aligned} \quad (4.4)$$

where we defined

$$g(\epsilon) := \exp \left\{ \left(\frac{8c_{11}^2}{d-1} + \frac{c_{11}+1}{c_{16}c_\Phi} \right) \epsilon^{-2(d-1)} \right\}, \text{ and } f(\epsilon) := 4e^{\frac{4}{\epsilon} + \frac{2}{d-1}} c_{16}c_\Phi \epsilon^{\frac{1}{2d^4}}.$$

The estimates in (4.4) hold because using $n! < n^n$,

$$\begin{aligned} (2\delta)^{-2m} ((2m)!)^{\frac{2}{d-1}} &\leq \exp \left\{ \frac{2}{d-1} 2m \ln(2m) - 2m \ln(2\delta) \right\} \\ &\leq \exp \left\{ \frac{8m^2}{d-1} + \frac{m}{\delta} \right\} \leq g(\epsilon) \end{aligned}$$

and, with Stirling's approximation $n! > n^n e^{-n}$ and the inequality $\frac{\ln n}{m} \leq \frac{1}{\epsilon}$, we have

$$\begin{aligned} n^{\frac{2}{d-1}} \left(n^{2m-d} (4\delta)^n (n!)^{-\frac{2}{d-1}} \right)^{1/n} &\leq n^{(2m-d)/n} (4\delta e^{\frac{2}{d-1}}) \\ &\leq \exp \left\{ 2m \frac{\ln n}{n} \right\} (4\delta e^{\frac{2}{d-1}}) \\ &\leq e^{\frac{4}{\epsilon}} 4c_{16}c_\Phi \epsilon^{\frac{1}{2d^4}} e^{\frac{2}{d-1}} = f(\epsilon) \end{aligned}$$

for $n > m^2$.

4.4 Proof of Theorem 4.3

The proof of Theorem 4.3 is based on the following strategy: we construct a set of polytopes with n facets and with bounded Φ -content which we obtain by slightly perturbing a deterministic polytope which is as *regular* as possible. We do so in a

way which ensures that Z is one of these polytopes with a high enough probability. In Lemma 4.6, we proceed with the construction of the deterministic polytope and in Lemma 4.7, we estimate the probability that Z is a perturbation of this deterministic polytope.

The arguments rely on a particular assumption on the directional distribution φ . A set $C \subset \mathbb{S}^{d-1}$ is called a cap of radius r if it is the intersection of \mathbb{S}^{d-1} with a ball of radius r having its center on the sphere \mathbb{S}^{d-1} . In the following we assume that φ is well spread and thus there is a cap C of radius $r < 1$ on the unit sphere and a constant c_{25} with

$$\varphi(\cdot) > c_{25} \mathcal{H}^{d-1}(\cdot).$$

Without loss of generality we assume that the cap is centred at the point $\mathbf{e}_d = (0, \dots, 0, 1)$. Observe that since φ is an even measure it is well spread on $C \cup (-C)$.

We start with two lemmata. The first one essentially ensures that all polyhedra occurring in this section are contained in a big ball and hence are bounded. The second lemma constructs sets $S_i \subset \mathcal{H}$ such that the outer normals of the corresponding halfspaces are in $C \cup (-C)$, their measure is of order $O(n^{-\frac{d+1}{d-1}})$, and their intersection forms a polytope with n facets in B^d . In the following we write $C(\mathbf{y}, \rho) = B(\mathbf{y}, \rho) \cap \mathbb{S}^{d-1}$ for caps on the sphere.

Lemma 4.6. *There exist a constant $c_{24} = c_{24}(d)$ and $m = m(d, r) < c_{24}$ points $\mathbf{y}_i \in C \cup (-C)$, $i = 1, \dots, m$ such that the caps $C(\mathbf{y}_i, r/12)$ are pairwise disjoint and*

$$\bigcap_{i=1}^m H(\mathbf{v}_i, 1)^- \subset B(\mathbf{o}, 4r^{-1})$$

for any $\mathbf{v}_i \in C(\mathbf{y}_i, r/12) \cap (C \cup (-C))$, $i = 1, \dots, m$.

Proof. We choose a saturated packing of caps $C(\mathbf{y}_i, r/12)$ with $\mathbf{y}_i \in C \cup (-C)$, $i = 1, \dots, m$. Here we call a packing saturated if there is no possibility for adding another ball of radius $r/12$. Since the curvature of the sphere becomes negligible when $r \rightarrow 0$, we have that m is of the same order as a saturated packing of $(d-1)$ -dimensional balls of radius $r/12$ in rB^{d-1} . Clearly this is independent from r and therefore $m < c_{24}$ for some constant c_{24} depending only on d .

This implies first that $\bigcup C(\mathbf{y}_i, r/6)$ is a covering of $C \cup (-C)$. Second, each cap $C(\mathbf{z}, r/4)$, $\mathbf{z} \in C$ contains one of the caps $C(\mathbf{y}_i, r/12)$, because $\mathbf{z} \in C(\mathbf{y}_i, r/6)$ for some $i = 1, \dots, m$.

The rest of the proof follows from explicit geometric calculations. Assume in the contrary that there are $\mathbf{v}_i \in C(\mathbf{y}_i, r/12) \cap (C \cup (-C))$ such that

$$\bigcap_{i=1}^m H(\mathbf{v}_i, 1)^- \not\subset B(\mathbf{o}, 4r^{-1}).$$

This in particular implies that either

$$\mathbf{e}_d^\perp \cap \bigcap_{\mathbf{v}_i \in C} H(\mathbf{v}_i, 1)^- \not\subset B(\mathbf{o}, 4r^{-1}) \quad \text{or} \quad \mathbf{e}_d^\perp \cap \bigcap_{\mathbf{v}_i \in -C} H(\mathbf{v}_i, 1)^- \not\subset B(\mathbf{o}, 4r^{-1}).$$

Recall that C is a cap with center \mathbf{e}_d . Without loss of generality assume that $\mathbf{x} = (4r^{-1}, 0, \dots, 0)$ is a point with $\|\mathbf{x}\| = 4r^{-1}$ which is contained in $\bigcap_{\mathbf{v}_i \in C} H(\mathbf{v}_i, 1)^-$. Let us define $\mathbf{x}_0 = (r/4, 0, \dots, 0, \sqrt{1 - r^2/16})$. By elementary trigonometric calculations the line through \mathbf{x} and \mathbf{x}_0 is tangent to the sphere at \mathbf{x}_0 . Because \mathbf{x} is contained in $\bigcap H(\mathbf{v}_i, 1)^-$, none of the points \mathbf{v}_i may be contained the cap $C_{\mathbf{x}} = C(\mathbf{e}_1, \|\mathbf{e}_1 - \mathbf{x}_0\|)$.

Next observe that the point $\mathbf{x}_C = (\sqrt{1 - h^2}, 0, \dots, 0, h)$ with $h = 1 - r^2/2$ is on the relative boundary of C and in $C_{\mathbf{x}}$, and

$$\|\mathbf{x}_C - \mathbf{x}_0\| \geq \sqrt{1 - h^2} - \frac{1}{4}r \geq \frac{3}{4}r - \frac{1}{4}r \geq \frac{1}{2}r.$$

Hence $C \cap C_{\mathbf{x}}$ contains a cap of radius $r/4$. Yet this cap must contain one of the caps $C(\mathbf{y}_i, r/12)$ and thus one of the points \mathbf{v}_i , a contradiction. \square

In the following lemma we assume that there exists a cap C of radius $r \in (0, 1)$ of the sphere and a constant c_{25} with $\varphi(\cdot) > c_{25}\mathcal{H}^{d-1}(\cdot)$ on C .

Lemma 4.7. *There exists a constant c_{29} such that the following holds. For any $n > c_{24}$, there are disjoint subsets $S_1, \dots, S_n \subset \mathcal{H}$ with*

$$\mu(S_i) > c_{25}c_{29}r^{d+2}n^{-\frac{d+1}{d-1}}$$

and for $H_1 \in S_1, \dots, H_n \in S_n$ we have

$$\bigcap_i H_i^- \in \mathcal{P}_n$$

and

$$\bigcap_i H_i^- \subset B^d. \quad (4.5)$$

Proof. Consider the $m < c_{24}$ caps $C(\mathbf{y}_i, r/12)$ which have been constructed in Lemma 4.6, and fix $n > c_{24}$. In each of the sets $C(\mathbf{y}_i, r/12) \cap (C \cup -C)$ we produce an optimal packing of $\lceil n/m \rceil$ smaller caps $C(\mathbf{z}_j, \rho)$ where we can choose ρ such that it satisfies

$$c_{30}n^{-\frac{1}{d-1}}r \leq \rho \leq \frac{r}{12}$$

with a constant c_{30} independent of r, n and m . Observe that the number of caps constructed in this way is between n and $n + m$. We choose precisely n of these caps $C(\mathbf{z}_i, \rho)$ in such a way that in each set $C(\mathbf{y}_i, r/12) \cap (C \cup -C)$ there is at least one cap $C(\mathbf{z}_i, \rho)$.

As already used above, a cap of radius t has height $t^2/2$. Let \mathbf{v}_i be arbitrary points in $C(\mathbf{z}_i, \rho/2)$, $i = 1, \dots, n$. Since each cap

$$C\left(\mathbf{v}_i, \frac{\rho}{2}\right) = H\left(\mathbf{v}_i, 1 - \frac{1}{2}\left(\frac{\rho}{2}\right)^2\right)^+ \cap \mathbb{S}^{d-1}$$

is contained in the cap $C(\mathbf{z}_i, \rho)$, it is disjoint from all other caps $C(\mathbf{z}_j, \rho)$, and thus also disjoint from all other caps $C(\mathbf{v}_j, \rho/2)$. Hence for arbitrary r_i with $0 \leq r_i \leq \rho/2$, all points $(1 - r_i^2/2) \mathbf{v}_i$ are on the boundary of $\cap_{i=1}^n H(\mathbf{v}_i, 1 - r_i^2/2)^-$ and thus this intersection has n facets.

Since each set $C(\mathbf{y}_i, r/12)$ contains a cap $C(\mathbf{z}_i, \rho)$, there are m points $\mathbf{v}_i, \mathbf{v}_1, \dots, \mathbf{v}_m$ say, which belong to $C(\mathbf{y}_1, r/12), \dots, C(\mathbf{y}_m, r/12)$ respectively. Combining Lemma 4.6 applied to $\mathbf{v}_1, \dots, \mathbf{v}_m$ and the considerations above, we obtain: there are pairwise disjoint sets

$$T_i = \left\{ H(\mathbf{v}, t) : \mathbf{v} \in C\left(\mathbf{z}_i, \frac{\rho}{2}\right), t \in \left[1 - \frac{1}{2} \left(\frac{\rho}{2}\right)^2, 1\right] \right\} \subset \mathcal{H}, \quad i = 1, \dots, n,$$

such that for an arbitrary n -tuple $H(\mathbf{v}_i, t_i) \in T_i, i = 1, \dots, n$, we have

$$\bigcap_{i=1}^n H(\mathbf{v}_i, t_i)^- \subset \bigcap_{i=1}^m H(\mathbf{v}_i, t_i)^- \subset B(\mathbf{o}, 4r^{-1}) \quad \text{and} \quad \bigcap_{i=1}^n H(\mathbf{v}_i, t_i)^- \in \mathcal{P}_n.$$

We normalize such that $B(\mathbf{o}, 4r^{-1})$ is replaced by the unit ball and define

$$S_i = \left\{ H(\mathbf{v}, t) : \mathbf{v} \in C\left(\mathbf{z}_i, \frac{\rho}{2}\right), t \in \frac{r}{4} \left[1 - \frac{1}{2} \left(\frac{\rho}{2}\right)^2, 1\right] \right\} = \frac{r}{4} T_i \subset \mathcal{H}$$

for $i = 1, \dots, n$. The sets S_i have measure at least

$$\mu(S_i) \geq c_{25} \mathcal{H}^{d-1} \left(C\left(\mathbf{z}_i, \frac{\rho}{2}\right) \right) \frac{r \rho^2}{32} \geq c_{25} c_{29} r^{d+2} n^{-\frac{d+1}{d-1}}$$

since $\rho \geq c_{30} n^{-\frac{1}{d-1}} r$, where c_{29} is a constant depending only on d . This yields n sets S_i with the desired properties. \square

We point out that because of Lemma 4.7 and condition (4.5) therein, having $H_{\sigma(1)} \in S_1, \dots, H_{\sigma(n)} \in S_n$ for some permutation $\sigma \in \mathfrak{S}_n$ implies that $\Phi(\cap_{i=1}^n H_i^-) < 1$, $\mathfrak{c}(\cap_{i=1}^n H_i^-) \in B^d$ and $\cap_{i=1}^n H_i^- \in \mathcal{P}_n$. Using this and (2.8) yields

$$\begin{aligned} \mathbb{P}(f(Z) = n) &= \frac{\gamma^d}{\gamma^{(d)} \kappa_d} \frac{(n-d)!}{n!} \int_{\tilde{\mathcal{H}}^n} \mathbb{1}(\mathfrak{c}(\cap_{i=1}^n H_i^{\epsilon_i}) \in B^d) \mathbb{1}(\Phi(\cap_{i=1}^n H_i^{\epsilon_i}) < 1) \\ &\quad \mathbb{1}(\cap_{i=1}^n H_i^{\epsilon_i} \in \mathcal{P}_n) \, d\tilde{\mu}^n(\mathbf{H}^\epsilon), \\ &\geq \frac{\gamma^d}{\gamma^{(d)} \kappa_d} (n-d)! \int_{\mathcal{H}^n} \mathbb{1}(H_1 \in S_1) \cdots \mathbb{1}(H_n \in S_n) \, d\mu^n(\mathbf{H}) \\ &= \frac{\gamma^d}{\gamma^{(d)} \kappa_d} (n-d)! \mu(S_1) \cdots \mu(S_n) \\ &> \frac{\gamma^d}{\gamma^{(d)} \kappa_d} (n-d)! \left(c_{25} c_{29} r^{d+2} n^{-1-\frac{2}{d-1}} \right)^n \end{aligned}$$

for $n \geq c_{24}$. Stirling's approximation $n! > n^n e^{-n}$ implies

$$\frac{(n-d)!}{n^n} > \frac{n!}{n^{n+d}} > e^{-(d+1)n}$$

With $c_{23} = \min\left(1, \frac{1}{\kappa_d}\right) c_{29} e^{-(d+1)}$, this implies immediately the statement of Theorem 4.3.

5 Big Cells

In this section we are interested in the behaviour of the typical cell Z when $\Phi(Z)$ tends to infinity. In particular we aim at proving results on the asymptotic behaviour of $\mathbb{P}(\Phi(Z) > a)$ (Theorem 1.4), on the shape of such big cells in the general case (Theorem 1.5) and on the existence of a limit shape in the particular case when φ is concentrated on a finite set of points (Theorem 1.6).

To get Theorem 1.5, we need a new upper-bound for the probability of the event $\{\Phi(Z) > a\}$ intersected with the event that the cell is $(\varepsilon: i, j)$ -elongated, which is given below.

Theorem 5.1. *Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. There exist constants c_{31} and c_{32} , such that for any $\epsilon < c_{31}$ and for any $a \geq \gamma^{-1} \varepsilon^{-(2d+3)}$,*

$$\mathbb{P}\left(\Phi(Z) > a, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon\right) \leq \exp\left(-\gamma a + c_{32} \epsilon^{\frac{1}{6d^4}} (\gamma a)^{\frac{d-1}{d+1}}\right).$$

Actually, the bound in Theorem 5.1 is close to the upper-bound from Theorem 1.4 but with a constant in front of $(\gamma a)^{\frac{d-1}{d+1}}$ which is arbitrarily small.

In the next subsection, we show how to deduce easily Theorem 1.5 from Theorems 1.4 and 5.1. The rest of Section 5 is devoted to the proof of Theorems 1.4, 5.1 and 1.6.

5.1 Deducing Theorem 1.5 from Theorems 1.4 and 5.1

Let us assume that φ is well spread and that $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. Then the lower-bound from Theorem 1.4 together with Theorem 5.1 imply that for any $\epsilon < c_{31}$ and for any $a \geq \gamma^{-1} \max\{c_5, \varepsilon^{-(2d+3)}\}$, we have

$$\mathbb{P}\left(\frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \mid \Phi(Z) > a\right) \leq \exp\left(\left(c_{32} \epsilon^{\frac{1}{6d^4}} - c_4\right) (\gamma a)^{\frac{d-1}{d+1}}\right).$$

For $\epsilon < \left(\frac{c_4}{c_{32}}\right)^{6d^4}$, the conditional probability above goes to zero when a goes to ∞ . This shows Theorem 1.5.

5.2 Proof of Theorem 1.4

We start with three intermediary lemmas: Lemma 5.2 builds upon the Complementary Theorem to get a rewriting of the distribution tail of $\Phi(Z)$ as a function of the distribution tail of $f(Z)$. In Lemma 5.3, we deduce from Theorems 1.1 and 1.2 respectively upper and lower-bounds for the distribution tail of $f(Z)$. Finally, Lemma 5.4 contains analytical estimates for some subexponential power series.

In the sequel, we use the abbreviations $q_n := \mathbb{P}(f(Z) = n)$ and $r_n := \sum_{k \geq n} q_k$ for every $n \geq (d+1)$.

In the following lemma, we rewrite the probability $\mathbb{P}(\Phi(Z) > a)$ as a power series in a .

Lemma 5.2. *For every $a > 0$, we have*

$$\mathbb{P}(\Phi(Z) > a) = e^{-\gamma a} \sum_{n \geq 0} r_{n+d+1} \frac{(\gamma a)^n}{n!}$$

Proof. Because of the Complementary Theorem 2.2 we have for every $a > 0$

$$\begin{aligned} \mathbb{P}(\Phi(Z) > a) &= \sum_{n \geq d+1} q_n \mathbb{P}(\Phi(Z) > a \mid f(Z) = n) \\ &= \sum_{n \geq d+1} q_n \int_a^\infty e^{-\gamma t} \frac{\gamma^{n-d} t^{n-d-1}}{(n-d-1)!} dt. \end{aligned}$$

Now we recall that iterated integrations by parts show that for every $n \geq (d+1)$,

$$\int_a^\infty e^{-\gamma t} \frac{\gamma^{n-d} t^{n-d-1}}{(n-d-1)!} dt = e^{-\gamma a} \sum_{m=0}^{n-d-1} \frac{(\gamma a)^m}{m!}.$$

Consequently, we obtain that

$$\mathbb{P}(\Phi(Z) > a) = e^{-\gamma a} \sum_{n \geq d+1} \sum_{m=0}^{n-d-1} q_n \frac{(\gamma a)^m}{m!} = e^{-\gamma a} \sum_{m \geq 0} r_{m+d+1} \frac{(\gamma a)^m}{m!},$$

which shows Lemma 5.2. □

The relation from Lemma 5.2 indicates that in order to bound $\mathbb{P}(\Phi(Z) > a)$, we need to find bounds for r_{n+d+1} . This is done in the next lemma.

Lemma 5.3. *There exists a constant c_{33} depending on φ such that for $n \geq 0$ we have*

$$r_{n+d+1} < c_{33}^n (n!)^{-\frac{2}{d-1}}.$$

Assume that φ is well spread. Then there exists a constant $c_{34} > 0$ depending on φ such that for $n \geq 0$ we have

$$r_{n+d+1} \geq c_{34}^n (n!)^{-\frac{2}{d-1}}.$$

Proof. We start with the upper-bound. By Theorem 1.1 we have for $n \geq d + 1$,

$$q_n < c_1^n n^{-\frac{2n}{d-1}}. \quad (5.1)$$

with some constant $c_1 > 0$ depending on φ . By (5.1) we have,

$$\begin{aligned} r_{n+d+1} &\leq \sum_{k \geq n+d+1} c_1^k k^{-\frac{2}{d-1}k} \\ &\leq c_1^n n^{-\frac{2}{d-1}n} \sum_{k \geq d+1} c_1^k k^{-\frac{2}{d-1}k}. \end{aligned}$$

We use $n^{-n} \leq (n!)^{-1}$, and observe that the remaining sum is convergent and independent of n . Hence in order to get the upper-bound, it suffices to set

$$c_{33} := c_1 \max \left\{ 1, \sum_{k \geq d+1} c_1^k k^{-\frac{2}{d-1}k} \right\}.$$

We assume now that φ is well spread and prove the lower-bound for r_{n+d+1} . Theorem 1.2 tells us that when φ is well spread, for every $n \geq 0$,

$$q_{n+d+1} > c_2^{n+d+1} (n+d+1)^{-\frac{2(n+d+1)}{d-1}}$$

Consequently, using Stirling's approximation $n^{-n} > e^{-n}(n!)^{-1}$ and the simple inequality $r_{n+d+1} > q_{n+d+1}$, we get

$$\begin{aligned} r_{n+d+1} &> \left(c_2 e^{-\frac{2}{d-1}} \right)^{n+d+1} [(n+d+1)!]^{-\frac{2}{d-1}} \\ &> \left(c_2 e^{-\frac{2}{d-1}} \right)^{n+d+1} [(n+d+1)^{d+1} \cdot n!]^{-\frac{2}{d-1}} \\ &> \left(c_2 (d+1)^{-\frac{2}{d-1}} e^{-\frac{2}{d-1}} \right)^{n+d+1} (n!)^{-\frac{2}{d-1}} \end{aligned}$$

because $(n+d+1)^{d+1} < (d+1)^{n+d+1}$ for $n+d+1 \geq d+1 \geq 3$.

Taking $c_{34} = c_2 (d+1)^{-\frac{2}{d-1}} e^{-\frac{2}{d-1}} \min(1, (c_2 (d+1)^{-\frac{2}{d-1}} e^{-\frac{2}{d-1}})^{d+1})$, we get the required result. \square

The combination of the two previous lemmas implies that $\mathbb{P}(\Phi(Z) > a)$ is well approximated by subexponential power series of type $\sum_{n \geq 0} \frac{x^n}{(n!)^\alpha}$. The next lemma, which is purely analytical, investigates the behaviour of such power series.

Lemma 5.4. *For any $\alpha > 1$, we have*

$$\exp \left(\frac{1}{2} \alpha x^{\frac{1}{\alpha}} \right) < \sum_{n \geq d+1} \frac{x^n}{(n!)^\alpha} < \sum_{n \geq 0} \frac{x^n}{(n!)^\alpha} < \exp \left(\alpha x^{\frac{1}{\alpha}} \right)$$

where the first inequality holds for $x \geq (2(3d+5))^\alpha$ and the second for all $x > 0$.

Proof. The right hand side inequality follows immediately from the following simple computations.

$$\sum_{n \geq d+1} \frac{x^n}{(n!)^\alpha} < \sum_{n \geq 0} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha < \left(\sum_{n \geq 0} \frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha = \exp \left(\alpha x^{\frac{1}{\alpha}} \right).$$

For the left hand side inequality, Hölder's inequality gives for any finite $I \subset \mathbb{N} \setminus [d+1]$

$$\sum_{n \geq d+1} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha \geq \sum_{n \in I} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha \geq |I|^{-(\alpha-1)} \left(\sum_{n \in I} \frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha. \quad (5.2)$$

For Y a Poisson distributed random variable with mean λ it is well known, and can be proved e.g. by Chebishev's inequality, that for $I = (\lambda - \sqrt{2\lambda}, \lambda + \sqrt{2\lambda}) \cap \mathbb{N}$, we have

$$\sum_{n \in I} e^{-\lambda} \frac{\lambda^n}{n!} = 1 - \mathbb{P}(|Y - \lambda| \geq \sqrt{2\lambda}) \geq \frac{1}{2}.$$

I has at most $2\sqrt{2\lambda} + 1 < 4\sqrt{\lambda}$ elements, when $\lambda \geq 1$. Putting this for $\lambda = x^{1/\alpha}$ into (5.2) yields

$$\sum_{n \geq d+1} \left(\frac{(x^{\frac{1}{\alpha}})^n}{n!} \right)^\alpha \geq \left(4x^{\frac{1}{2\alpha}} \right)^{-(\alpha-1)} \left(e^{x^{\frac{1}{\alpha}}} \frac{1}{2} \right)^\alpha \geq \left(8^{-\alpha} x^{-\frac{1}{2}} \right) e^{\alpha x^{\frac{1}{\alpha}}}$$

as long as the condition $x^{1/\alpha} - \sqrt{2x^{1/\alpha}} \geq d+1$ is fulfilled. Observe that $x \geq (3d+5)^\alpha$ implies $x^{1/(2\alpha)} \geq \sqrt{d+2} + 1$ which in turn implies $x^{1/\alpha} - 2x^{1/(2\alpha)} + 1 \geq d+2$ which gives the required condition.

For $t \geq 3$ we have $2 \ln 8 + t \leq 1 + t + t^2/2 \leq e^t$, or equivalently

$$-\alpha \ln 8 - \frac{1}{2} \ln x \geq -\frac{1}{2} \alpha x^{\frac{1}{\alpha}}, \quad \text{i.e.} \quad 8^{-\alpha} x^{-\frac{1}{2}} \geq e^{-\frac{1}{2} \alpha x^{1/\alpha}}$$

for $x^{1/\alpha} \geq e^3$. The inequality $2(3d+5) > e^3$ concludes the proof. \square

We are now ready to prove Theorem 1.4. Combining Lemma 5.2 and the upper-bound of Lemma 5.3, we get

$$\mathbb{P}(\Phi(Z) > a) < e^{-\gamma a} \sum_{n \geq 0} c_{33}^n \frac{(\gamma a)^n}{(n!)^{\frac{d+1}{d-1}}}.$$

Applying now Lemma 5.4 to $x = c_{33}\gamma a$ and $\alpha = \frac{d+1}{d-1}$, we obtain that

$$\mathbb{P}(\Phi(Z) > a) < e^{-\gamma a} \sum_{n \geq 0} \frac{(c_{33}\gamma a)^n}{(n!)^{\frac{d+1}{d-1}}} < \exp \left(-\gamma a + \frac{d+1}{d-1} (c_{33}\gamma a)^{\frac{d-1}{d+1}} \right).$$

The proof of the lower-bound is nearly identical and we leave the details to the reader.

5.3 Proof of Theorem 5.1

Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. In the sequel, we use the notation $q_n^\varepsilon := \mathbb{P} \left(f(Z) = n, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \varepsilon \right)$ and $r_n^\varepsilon := \sum_{k \geq n} q_k^\varepsilon$, for every $n \geq (d+1)$ and $\varepsilon > 0$. The proof follows along the same lines as the upper bound of Theorem 1.4 with minor adaptations. Indeed, we need some analogues to the statements of Lemmas 5.2 and 5.3 when q_n is replaced by q_n^ε , i.e. when the extra-condition that Z is $(\varepsilon: i, j)$ -elongated is added.

The lemma below is a rewriting of the joint distribution of $(\mathfrak{s}(Z), \Phi(Z))$ as a power series.

Lemma 5.5. *For any measurable set of shapes $S \subset \mathcal{K}_{\mathfrak{c}, \Phi}$ and $a > 0$, we have*

$$\begin{aligned} \mathbb{P}(\mathfrak{s}(Z) \in S, \Phi(Z) > a) &= e^{-\gamma a} \sum_{k \geq d+1} \mathbb{P}(\mathfrak{s}(Z) \in S, f(Z) = k) \sum_{l=0}^{k-d-1} \frac{(\gamma a)^l}{l!} \\ &= e^{-\gamma a} \sum_{l \geq 0} \mathbb{P}(\mathfrak{s}(Z) \in S, f(Z) \geq l + d + 1) \frac{(\gamma a)^l}{l!}. \end{aligned}$$

The proof of this result is fully analogous to that of Lemma 5.2 and is therefore omitted.

As in Lemma 5.3, we require now an upper-bound for r_{n+d+1}^ε .

Lemma 5.6. *Assume $1 \leq i < j \leq \lceil (d-1)/2 \rceil$. There exist constants c_{31} and c_{35} depending on φ , such that for any $\varepsilon < c_{31}$ we have*

$$r_{n+d+1}^\varepsilon < e^{c_{21}\varepsilon^{-2(d-1)}} (c_{35}\varepsilon^{\frac{1}{2d^4}})^n (n!)^{-\frac{2}{d-1}}$$

for $n \geq 0$.

Proof. Theorem 4.2 implies that for any $\varepsilon < c_{11}^{2/(d-1)} c_{12}^{-1} (c_\Phi)^{-1}$ and $n > \lfloor c_{11}\varepsilon^{-(d-2)} \rfloor^2$,

$$q_n^\varepsilon < \frac{\gamma^d}{\gamma^{(d)}} e^{c_{21}\varepsilon^{-2(d-1)}} (c_{22}\varepsilon^{\frac{1}{2d^4}})^n n^{-\frac{2n}{d-1}}.$$

For $\varepsilon < c_{31}$ we have

$$\begin{aligned} r_{n+d+1}^\varepsilon &\leq \frac{\gamma^d}{\gamma^{(d)}} e^{c_{21}\varepsilon^{-2(d-1)}} \sum_{k \geq n+d+1} \left(c_{22}\varepsilon^{\frac{1}{2d^4}} \right)^k k^{-\frac{2}{d-1}k} \\ &\leq \frac{\gamma^d}{\gamma^{(d)}} e^{c_{21}\varepsilon^{-2(d-1)}} (c_{22}\varepsilon^{\frac{1}{2d^4}})^n n^{-\frac{2n}{d-1}} \sum_{k \geq d+1} \left(c_{22}c_{31}^{\frac{1}{2d^4}} \right)^k k^{-\frac{2}{d-1}k}. \end{aligned}$$

We use $n^{-n} \leq (n!)^{-1}$, and observe that the remaining sum is convergent and independent of n . Hence it suffices to set

$$c_{35} := c_{22} \max \left\{ 1, \sum_{k \geq d+1} \left(c_{22}c_{31}^{\frac{1}{2d^4}} \right)^k k^{-\frac{2}{d-1}k} \right\}.$$

□

Let us now proceed with the proof of Theorem 5.1. Applying Lemma 5.5 to the set $S = \{K \in \mathcal{K}_{\mathbf{c}, \Phi} : V_j(Z)^{1/j} V_i(Z)^{-1/i} < \epsilon\}$, we get

$$\mathbb{P} \left(\Phi(Z) > a, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \right) = e^{-\gamma a} \sum_{n \geq 0} r_{n+d+1}^\epsilon \frac{(\gamma a)^n}{n!}.$$

We combine this with Lemma 5.6 to deduce

$$\mathbb{P} \left(\Phi(Z) > a, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \right) \leq e^{-\gamma a + c_{21} \epsilon^{-2(d-1)}} \sum_{n \geq 0} (c_{35} \epsilon^{\frac{1}{2d^4}})^n \frac{(\gamma a)^n}{(n!)^{\frac{d+1}{d-1}}}.$$

Lemma 5.4 ends the proof:

$$\begin{aligned} \mathbb{P} \left(\Phi(Z) > a, \frac{V_j(Z)^{\frac{1}{j}}}{V_i(Z)^{\frac{1}{i}}} < \epsilon \right) &\leq \exp \left(-\gamma a + \frac{d+1}{d-1} (c_{35} \epsilon^{\frac{1}{2d^4}} \gamma a)^{\frac{d-1}{d+1}} + c_{21} \epsilon^{-2(d-1)} \right) \\ &\leq \exp \left(-\gamma a + c_{32} \epsilon^{\frac{1}{6d^4}} (\gamma a)^{\frac{d-1}{d+1}} \right) \end{aligned}$$

for $\epsilon^{-(2d+3)} \leq \gamma a$ because this implies $\epsilon^{-2(d+1)} \leq \epsilon \gamma a \leq \epsilon^{\frac{1}{2d^4}} \gamma a$ and thus $\epsilon^{-2(d-1)} \leq (\epsilon^{\frac{1}{2d^4}} \gamma a)^{\frac{d-1}{d+1}} \leq \epsilon^{\frac{1}{6d^4}} (\gamma a)^{\frac{d-1}{d+1}}$ since $\frac{d-1}{d+1} \geq \frac{1}{3}$.

5.4 Proof of Theorem 1.6

Assume φ is concentrated on a finite number n_{\max} of points. Thus $f(Z) \leq n_{\max}$ with probability one. We use again the notation $q_n = \mathbb{P}(f(Z) = n)$, and fix some subset $S \subset \mathcal{K}_{\mathbf{c}, \Phi}$ of the shape space such that $\mathbb{P}(\mathfrak{s}(Z) \in S, f(Z) = n_{\max}) > 0$. Because of Lemma 5.5, we have

$$\begin{aligned} \mathbb{P}(\mathfrak{s}(Z) \in S, \Phi(Z) > a) &= e^{-\gamma a} \sum_{k \leq n_{\max}} \mathbb{P}(\mathfrak{s}(Z) \in S, f(Z) = k) \sum_{l=0}^{k-d-1} \frac{(\gamma a)^l}{l!} \\ &= e^{-\gamma a} \mathbb{P}(\mathfrak{s}(Z) \in S, f(Z) = n_{\max}) \sum_{l=0}^{n_{\max}-d-1} \frac{(\gamma a)^l}{l!} (1 + O((\gamma a)^{-1})) \\ &= \mathbb{P}(\mathfrak{s}(Z) \in S, \Phi(Z) \geq a, f(Z) = n_{\max}) (1 + O((\gamma a)^{-1})) \end{aligned}$$

This implies

$$\begin{aligned} \mathbb{P}(\mathfrak{s}(Z) \in S | \Phi(Z) \geq a) &= \frac{\mathbb{P}(\mathfrak{s}(Z) \in S, \Phi(Z) \geq a)}{\mathbb{P}(\Phi(Z) \geq a)} \\ &= \frac{\mathbb{P}(\mathfrak{s}(Z) \in S, \Phi(Z) \geq a, f(Z) = n_{\max}) (1 + O((\gamma a)^{-1}))}{\mathbb{P}(\Phi(Z) \geq a, f(Z) = n_{\max}) (1 + O((\gamma a)^{-1}))} \\ &= \frac{\mathbb{P}(\mathfrak{s}(Z) \in S, \Phi(Z) \geq a | f(Z) = n_{\max})}{\mathbb{P}(\Phi(Z) \geq a | f(Z) = n_{\max})} (1 + O((\gamma a)^{-1})) \\ &= \mathbb{P}(\mathfrak{s}(Z) \in S | f(Z) = n_{\max}) (1 + O((\gamma a)^{-1})) \end{aligned}$$

where in the last equation we used again the Complementary Theorem 2.2.

References

- [1] Bonnet, G.: Polytopal approximation of elongated convex bodies. *Adv. Geom.*, to appear
- [2] Calka, P.: Tessellations. In *New perspectives in stochastic geometry*. Oxford Univ. Press, Oxford, 145–169
- [3] Calka, P.: Asymptotic methods for random tessellations. In *Stochastic geometry, spatial statistics and random fields, Lecture Notes in Math.* **2068**, Springer, Heidelberg, 183–204
- [4] Calka, P., Hilhorst, H. J.: Random line tessellations of the plane: statistical properties of many-sided cells. *J. Stat. Phys.* **132** (2008), 627–647
- [5] Cowan, R.: A more comprehensive complementary theorem for the analysis of Poisson point processes. *Adv. Appl. Prob.* **38** (2006), 581–601
- [6] Gruber, P. M.: *Convex and discrete geometry, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **336**, Springer, Berlin (2007)
- [7] Hilhorst, H. J.: Heuristic theory for many-faced d -dimensional Poisson-Voronoi cells. *J. Stat. Mech.* (2009), P08003
- [8] Hug, D.: Random mosaics. In *Stochastic geometry, Lecture Notes in Math.* **1892**. Springer, Berlin, 247–266
- [9] Hug, D., Reitzner, M.: Introduction to Stochastic Geometry. In *Stochastic analysis for Poisson point processes: Malliavin calculus, Wiener-It chaos expansions and stochastic geometry*. Bocconi & Springer Series **7**. Bocconi University Press, Springer, Milano (2016), 145–184
- [10] Hug, D., Reitzner, M., Schneider, R.: The limit shape of the zero cell in a stationary Poisson hyperplane tessellation. *Ann. Probab.* **32** (2004), 1140–1167
- [11] Hug, D., Schneider, R.: Asymptotic shapes of large cells in random tessellations. *Geom. Funct. Anal.* **17** (2007), 156–191
- [12] Hug, D., Schneider, R.: Typical cells in Poisson hyperplane tessellations. *Discrete Comput. Geom.* **38** (2007), 305–319
- [13] Hug, D., Schneider, R.: Large faces in Poisson hyperplane mosaics. *Ann. Probab.* **38** (2010), 1320–1344
- [14] Kovalenko, I.: A proof of a conjecture of David Kendall on the shape of random polygons of large area. (Russian). *Kibernet. Sistem. Anal.* **187** (1997), 3–10. Engl. transl.: *Cybernet. Systems Anal.* **33**, 461–467, 1997.

- [15] Matheron, G.: *Random sets and integral geometry*, Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York-London-Sydney (1975)
- [16] Miles, R. E.: Poisson flats in Euclidean spaces. II. Homogeneous Poisson flats and the complementary theorem. *Adv. Appl. Probab.* **3** (1971), 1–43.
- [17] Miles, R. E.: The random division of space. *Suppl. Adv. Appl. Probab.* **4** (1972), 243–266.
- [18] Miles, R. E.: The various aggregates of random polygons determined by random lines in a plane. *Adv. in Math.* **10** (1973), 256–290.
- [19] Møller, J., Zuyev, S.: Gamma-type results and other related properties of Poisson processes. *Adv. Appl. Probab.* **28** (1996), 662–673.
- [20] Reisner, S., Schütt, C., Werner, E.: Dropping a vertex or a facet from a convex polytope. *Forum Math.* **13** (2001), 359–378
- [21] Schneider, R.: *Convex Bodies: The Brunn-Minkowski Theory, Second expanded edition. Encyclopedia of Mathematics and Its Applications.* **151** Cambridge University Press, Cambridge (2014)
- [22] Schneider, R., Weil, W.: *Stochastic and integral geometry.* Probability and its Applications (New York). Springer-Verlag, Berlin (2008)